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MINIMAL STRUCTURES, *m*-OPEN MULTIFUNCTIONS AND BITOPOLOGICAL SPACES

TAKASHI NOIRI AND VALERIU POPA

ABSTRACT : By using *m*-open multifunctions from a topological space into an *m*-space, we establish the unified theory for several weak forms of open multifunctions between bitopological spaces.

Key words and pharases : *m*-structure; *m*-open set, (*i*, *j*)-*m*-open multifunction, bitopological space, multifunction.
AMS Subject Classification : 54*A*05, 54*C*10, 54*C*60, 54*E*55.

1. INTRODUCTION

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of open functions and open multifunctions. By using these sets, several authors introduced and studied various types of modifications of open functions and open multifunctions in topological spaces and bitopological spaces. Maheshwari and Prasad [20] and Bose [6] introduced the concepts of semi-open sets and semi-open functions in bitopological spaces.. Jelic′ [12], [14], Kar and Bhattacharyya [15] and Khedr et al. [16] introduced and studied the concepts of preopen sets and preopen functions in bitopological spaces. The notions of α-open sets and α-open functions in bitopological spaces were studied in [13], [24] and [17]. Some forms of open multifunctions are studied in [5], [7] and [8]. Recently, in [30] and [31] the present authors introduced the notions of minimal structures, *m*-spaces and *m*-continuity.

In the present paper, we introduce the notion of an *m*-open multifunctions from a topological space into an *m*-space and establish the unified theory for several weak forms of open multifunctions between bitopological spaces. We obtain some characterizations of *m*-open multifunctions and characterize the set of all points at which a multifunction is not *m*-open. In the last part, some new modifications of open multifunctions between bitopological spaces is introduced and investigated.

2. PRELIMINARIES

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be *α-open* [25] (resp. *semi-open* [18], *preopen* [22], *β-open* [1] or *semi-preopen* [3], *b-open* [4] or *γ-open* [11]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$).

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) set is said to be *semi-closed* [9] (resp. *preclosed* [10], *α-closed* [23], *β-closed* [1], *semi-preclosed* [3], *b-closed* [4]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-preclosed, b-closed) sets of X containing A is called the *semi-closure* [9] (resp. *pre-closure* [10], *α-closure* [23], *β-closure* [2], *semi-preclosure* [3], *b-closure* [4]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$, $\mathrm{bCl}(A)$.

Definition 2.4. The union of all semi-open (resp. prepen, α-open, β-open, semi-preopen, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior, α-interior, β-interior, semi-preinterior, b-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$, $\mathrm{spInt}(A)$, $\mathrm{bInt}(A)$).

Throught the present paper, (X, τ) and (Y, σ) (briefly X and Y) always denote topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A function $f : (X, \tau) \to (Y, \tau)$ is said to be *open* at a point $x \in X$ if for each open set U containing x, there exists an open set V of Y containing $f(x)$ such that $V \subset f(U)$. If f is open at each point $x \in X$, then f is said to be *open.*

Remark 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if $f(U)$ is open for each open set U of X.

Definition 2.6. function $f : (X, \tau) \to (Y, \sigma)$ is said to be *semi-open* [26] (resp. *preopen* [22], *α-open* [23], *β-open* [1]) if $f(U)$ is semi-open (resp. preopen, α-open, β-open) for each open set U of X.

Definition 2.7. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be *open* [5] (resp. *semi-open* [29], *preopen* [8], *α-open* [7], *β-open*) if $F(U)$ is open (resp. semi-open, preopen, α-open, β-open) for each open set U of X.

3. MINIMAL STRUCTURES AND m-OPEN MULTIFUNCITONS

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [30], [31] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [21] as follows:

(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.1. Let (X, τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$, BO(X)), then we have

(a) $m_X\text{-Cl}(A) = \text{Cl}(A)$ (resp. sCl(A), pCl(A), αCl(A, βCl(A), bCl(A)),

(b) $m_X\text{-Int}(A) = \text{Int}(A)$ (resp. sInt(A), pInt(A), αInt(A, βInt(A), bInt(A)).

Lemma 3.1. (Maki et al. [21]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $m_X\text{-Cl}(X - A) = X - m_X\text{-Int}(A)$ *and* $m_X\text{-Int}(X - A) = X - m_X\text{-Cl}(A)$,

(2) *If* $(X - A) \in m_X$, *then* $m_X\text{-Cl}(A) = A$ *and if* $A \in m_X$, *then* $m_X\text{-Int}(A) = A$,

(3) $m_X\text{-Cl}(\emptyset) = \emptyset$, $m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\emptyset) = \emptyset$ *and* $m_X\text{-Int}(X) = X$,

(4) *If* $A \subset B$, *then* $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ *and* $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,

(5) $A \subset m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(A) \subset A$,

(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 3.2. (Popa and Noiri [30]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [21] if the union of any family of subsets belonging to m_X belongs to m_X.

Definition 3.3. (Popa and Noiri [32]). *Let (X, m_X) be an m-space and m_X satisfy property $\mathcal{B}$. Then for a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $m_X\text{-Int}(A) = A$,

(2) *A is m-closed if and only if* $m_X\text{-Cl}(A) = A$,

(3) m_X-Int$(A) \in m_X$ *and* m_X-Cl(A) *is* m_X*-closed.*

Remark 3.2. Let (X, τ) be a topological space and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$), then m_X satisfies property $\mathscr{B}$.

Definition 3.4. Let (Y, m_Y) be an m-space.

(1) A multifunction $F : (X, \tau) \to (Y, m_Y)$ is said to be *m-open* at $x \in X$ if for each open set U containing x, there exists $V \in m_Y$ containing $F(x)$ such that $V \subset F(U)$. If F is m-open at each point $x \in X$, then F is said to be *m-open*.

(2) A function $f : (X\ \tau) \to (Y, m_Y)$ is said to be *m-open* at $x \in X$ if for each open set U containing x, there exists $V \in m_Y$ containing $f(x)$ such that $V \subset f(U)$. If f is m-open at each point $x \in X$, then f is said to be *m-open*.

Theorem 3.1. *A multifunction* $F : (X, \tau) \to (Y, m_Y)$ *is m-open at* $x \in X$, *where* m_Y *has property* $\mathscr{B}$, *if and only if for each open set U containing,* x, $x \in F^+(m_Y\text{-Int}(F(U)))$.

Proof. *Necessity.* Let U be an open set containing x. Then, there exists $V \in m_Y$ such that $F(x) \subset V \subset F(U)$ and hence $F(x) \subset m_Y\text{-Int}(F(U))$. Therefore, we obtain that $x \in F^+(m_Y\text{-Int}(F(U)))$.

Sufficiency. Suppose that $x \in F^+(m_Y\text{-Int}(F(U)))$ for each open set U containing x. Then $F(x) \subset m_Y\text{-Int}(F(U))$. Set $V = m_Y\text{-Int}(F(U))$, then by Lemma 3.3 $V \in m_Y$ and $F(x) \subset V \subset F(U)$. Therefore, F is m-open at x.

Theorem 3.2. *A multifunction* $F : (X, \tau) \to (Y, m_Y)$ *is m-open, where* (Y, m_Y) *has property B, if and only if* $F(U)$ *is* m_Y*-open for each open set U of X.*

Proof. *Necessity.* Let U be an open set of X and $x \in U$. Since F is m-open at $x \in X$, by Theorem 3.1 we have $F(x) \subset m_Y\text{-Int}(F(U))$ and by Lemma 3.1 $F(U) = m_Y\text{-Int}(F(U))$. By Lemma 3.3, $F(U)$ is m_Y-open.

Sufficiency. Let $x \in X$ and U be an open set of X containing x. Then we have $F(x) \subset F(U) = m_Y\text{-Int}(F(U))$. Therefore $x \in F^+(m_Y\text{-Int}(F(U))$. By Theorem 3.1, F is m-open at arbitrary point $x \in X$.

Remark 3.3. (a) If $F : (X, \tau) \to (Y, \sigma)$ is a multifunciton and $m_Y = \tau$ (resp. $\mathrm{SO}(Y)$, $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), we obtain Definition 2.7, that is, the definition of an open (resp. semi-open, preopen, α-open, β-open) multifunction, where $\mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$) is the family of all semi-open (resp. preopen, α-open, β-open) sets of Y.

(b) If $f : (X, \tau) \to (Y, \sigma)$ is a function and $m_Y = \sigma$ (resp. $\mathrm{SO}(Y)$, $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), we obtain Definition 2.6.

Theorem 3.3. *For a multifunction* $F : (X, \tau) \to (Y, m_Y)$, *where* m_Y *has property* $\mathscr{B}$, *the following properties are equivalent:*

(1) *F is m-open at x;*

(2) *If $x \in \mathrm{Int}(A)$ for $A \in \mathcal{P}(X)$, then $x \in F^+(m_Y\text{-}\mathrm{Int}(F(A)))$;*

(3) *$x \in \mathrm{Int}(F^+(B))$ for $B \in \mathcal{P}(Y)$, then $x \in F^+(m_Y\text{-}\mathrm{Int}(B))$;*

(4) *If $x \in F^-(m_Y\text{-}\mathrm{Cl}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in \mathrm{Cl}(F^-(B))$.*

Proof. (1) $\Rightarrow$ (2) : Let $A \in \mathcal{P}(X)$ and $x \in \mathrm{Int}(A)$. Then, there exists an open set U such that $x \in U \subset A$ and hence $F(x) \subset F(U) \subset F(A)$. Since F is m-open at x, by Theorem 3.1 and Lemma 3.1, we obtain $x \in F^+(m_Y\text{-}\mathrm{Int}(F(U))) \subset F^+(m_Y\text{-}\mathrm{Int}(F(A))))$.

(2) $\Rightarrow$ (3) : Let $B \in \mathcal{P}(Y)$ and $x \in \mathrm{Int}\ (F^+(B))$. Then, $x \in F^+(m_Y\text{-}\mathrm{Int}(F(F^+(B)))) \subset F^+(m_Y\text{-}\mathrm{Int}(B))$.

(3) $\Rightarrow$ (4) : Let $B \in \mathcal{P}(Y)$ and $x \notin \mathrm{Cl}(F^-(B))$. Then $x \in X - \mathrm{Cl}(F^-(B)) = \mathrm{Int}(X - F^-(B)) = \mathrm{Int}(F^+(Y - B))$. By (3) we have $x \in F^+(m_Y\text{-}\mathrm{Int}(Y-B)) = X - F^-(m_Y\text{-}\mathrm{Cl}(B))$. Hence, $x \notin F^-(m_Y\text{-}\mathrm{Cl}(B))$. Therefore, if $x \in F^-(m_Y\text{-}\mathrm{Cl}(B))$, then $x \in \mathrm{Cl}(F^-(B))$.

(4) $\Rightarrow$ (1) : Let U be any open set of X containing x and $B = Y - F(U)$. Since $\mathrm{Cl}(F^-(B)) = \mathrm{Cl}(F^-(Y - F(U))) = \mathrm{Cl}(X - F^+(F(U))) \subset X - \mathrm{Int}(U) = X - U$ and $x \in U$, we obtain that $x \notin \mathrm{Cl}(F^-(B))$. By (4), we have $x \notin F^-(m_Y\text{-}\mathrm{Cl}(B)) = F^-(m_Y\text{-}\mathrm{Cl}(Y - F(U))) = X - F^+(m_Y\text{-}\mathrm{Int}(F(U)))$. Therefore, $x \in F^+(m_Y\text{-}\mathrm{Int}(F(U)))$. By Theorem 3.1, F is m-open at x.

Theorem 3.4. *For a multifunciton $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) *F is m-open;*

(2) *$F(\mathrm{Int}(A)) \subset m_Y\text{-}\mathrm{Int}(F(A))$ for any subset A of X;*

(3) *$\mathrm{Int}(F^+(B)) \subset F^+(m_Y\text{-}\mathrm{Int}(B))$ for any subset B of Y;*

(4) *$F^-(m_Y\text{-}\mathrm{Cl}(B)) \subset \mathrm{Cl}(F^-(B))$ for any subset B of Y.*

Proof. (1) $\Rightarrow$ (2) : Let A be any subset of X and $x \in \mathrm{Int}(A)$. Since F is m-open at each $x \in A$, by Theorem 3.3 $F(x) \subset m_Y\text{-}\mathrm{Int}(F(A)$. Hence $F(\mathrm{Int}(A)) \subset m_Y\text{-}\mathrm{Int}(F(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. By (2), we have $F(\mathrm{Int}(F^+(B))) \subset m_Y\text{-}\mathrm{Int}(F(F^+(B))) \subset m_Y\text{-}\mathrm{Int}(B)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By (3), we have $X - \mathrm{Cl}(F^-(B)) = \mathrm{Int}(X - F^-(B)) = \mathrm{Int}(F^+(Y - B)) \subset F^+(m_Y\text{-}\mathrm{Int}(Y - B)) = X - F^-(m_Y\text{-}\mathrm{Cl}(B))$. Hence, $F^-(m_Y\text{-}\mathrm{Cl}(B)) \subset \mathrm{Cl}(F^-(\mathrm{B}))$.

(4) $\Rightarrow$ (1): Let U be any open set of X and $B = Y - F(U)$. By (4), we have $F^-(m_Y\text{-}\mathrm{Cl}(Y - F(U))) \subset \mathrm{Cl}(F^-(Y - F(U))$. Now, $F^-(m_Y\text{-}\mathrm{Cl}(Y - F(U))) = F^-(Y - m_Y\text{-}\mathrm{Int}(F(U))) = X - F^+(m_Y\text{-}\mathrm{Int}(F(U)))$. And also we have $\mathrm{Cl}(F^-(Y - F(U))) = \mathrm{Cl}(X - F^+(F(U))) \subset X - \mathrm{Int}(U) = X - U$. Therefore, we obtain $U \subset F^+(m_Y\text{-}\mathrm{Int}(F(U)))$ and hence $F(U) \subset m_Y\text{-}\mathrm{Int}\ (F(U))$. By Lemma 3.1, $F(U) = m_Y\text{-}\mathrm{Int}(F(U)$ and by Theorem 3.2, F is m-open.

Remark 3.4. *Let* $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_Y = \mathrm{SO}(X)$. Then by Theorem 3.4, we obtain the result established in Lemma 5.2 of [29].

For a multifunction $F : (X, \sigma) \to (Y, m_Y)$, we denote

$$D^0(F) = \{x \in X : F \textit{ is not m-open at } x\}.$$

Theorem 3.5. *For a multifunction* $F : (X, \tau) \to (Y, m_Y)$, *where* m_Y *has property* $\mathcal{B}$, *the following properties hold:*

$$\begin{aligned} D^0(F) &= \cup_{U \in \tau}\{U - F^-(m_Y\text{-Int}(F(U)))\} \\ &= \cup_{A \in \mathcal{P}(X)}\{\mathrm{Int}(A) - F^+(m_Y\text{-Int}(F(A)))\} \\ &= \cup_{B \in \mathcal{P}(y)}\{\mathrm{Int}(F^+(B)) - F^+(m_Y\text{-Int}(B))\} \\ &= \cup_{B \in \mathcal{P}(y)}\{F^-(m_Y\text{-Cl}(B) - \mathrm{Cl}(F^-(B))\}. \end{aligned}$$

Proof. Let $x \in D^0(F)$. Then, by Theorem 3.1, there exists an open set U_0 containing x such that $x \notin F^+(m_Y\text{-Int}(F(U_0)))$. Hence $x \in U_0 \cap (X \setminus F^+(m_Y\text{-Int}(F(U_0)))) = U_0 \setminus F^+(m_Y\text{-Int}(F(U_0))) \subset \cup_{U \in \tau}\{U - F^+(m_Y\text{-Int}(F(U)))\}$.

Conversely, let $x \in \cup_{U \in \tau}\{U - F^+(m_Y\text{-Int}(F(U)))\}$. Then there exists $U_0 \in \tau$ such that $x \in U_0 - F^+(m_Y\text{-Int}(F(U_0)))$. Therefore, by Theorem 3.1 $x \in D^0(F)$.

For the second equation, let $x \in D^0(F)$. Then, by Theorem 3.3, there exists $A_1 \in \mathcal{P}(X)$ such that $x \in \mathrm{Int}(A_1)$ and $x \notin F^+(m_Y\text{-Int}(F(A_1)))$. Therefore, $x \in \mathrm{Int}(A_1) - F^+(m_Y\text{-Int}(F(A_1))) \subset \cup_{A \in \mathcal{P}(x)}\{\mathrm{Int}(A) - F^+(m_Y\text{-Int}(F(A)))\}$.

Conversely, $x \in \cup_{A \in \mathcal{P}(x)}\{\mathrm{Int}(A) - F^+(m_Y\text{-Int}(F(A)))\}$. Then there exists $A_1 \in \mathcal{P}(X)$ such that $x \in \mathrm{Int}(A_1) - F^+(m_Y\text{-Int}(F(A_1)))$. By Theorem 3.3, $x \in D^0(f)$.

The other equations are similarly proved.

4. MINIMAL STRUCTURES AND BITOPOLOGICAL SPACES

Throughout the present paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces. For a subset A of X, the closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$. First, we shall recall some definitions of weak forms of open sets in a bitopological space.

Definition 4.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

1. *(i, j)-semi-open* [20] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,
2. *(i, j)-preopen* [12] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,
3. *(i, j)-α-open* [13] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,
4. *(i, j)-semi-preopen* [16] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

The family of (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open (i, j)-semi-preopen) sets of (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

Remark 4.1. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then $(i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$ and $(i, j)\mathrm{SPO}(X)$ are all m-structures on X. Hence, if $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$), then we have

(1) m_{ij}-$\mathrm{Cl}(A) = (i, j)$-$s\mathrm{Cl}(A)$ [20] (resp. (i, j)-$\mathrm{pCl}(A)$ [16], (i, j)-$\alpha\mathrm{Cl}(A)$ [24], (i, j)-$\mathrm{spCl}(A)$ [16]),

(2) m_{ij}-$\mathrm{Int}(A) = (i, j)$-$\mathrm{sInt}(A)$ (resp. (i, j)-$\mathrm{pInt}(A)$, (i, j)-$\alpha\mathrm{Int}(A)$, (i, j)-$\mathrm{spInt}(A)$).

Remark 4.2. Let (X, τ_1, τ_2) be a bitopological space.

(a) Let $m_i = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\alpha(X)$). Then, by Lemma 3.1 we obtain the result established in Theorem 13 of [20] and Theorem 1.13 of [19] (resp. Theorem 3.6 of [24]).

(b) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 3.2 we obtain the result established in Theorem 1.15 of [19] (resp. Theorem 3.5 of [16], Theorem 3.5 of [24], Theorem 3.6 of [16]).

Remark 4.3. Let (X, τ_1, τ_2) be a bitopological space.

(a) It follows from Theorem 2 of [20] (resp. Theorem 4.2 of [15] or theorem 3.2 of [16], Theorem 3.2 of [24], Theorem 3.2 of [16] that $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$) is an m-structure on X satisfying property $\mathcal{B}$.

(b) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 3.3 we obtain the result established in Theorem 1.13 of [19] (resp. Theorem 3.5 of [16], Theorem 3.6 of [24], Theorem 3.6 of [16]).

5. m-OPEN MULTIFUNCTIONS IN BITOPOLOGICAL SPACES

Definition 5.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-semi-open* [6] (resp. *(i, j)-preopen* [15], *(i, j)-α-open* [17], *(i, j)-semi-preopen*) if for each τ_i-open set U of X, $f(U)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-prepen) in Y.

Definition 5.2. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-almost open* (cr *(i, j)-preopen*) [8] if for each $U \in \tau_i$, $F(U)$ is (i, j)-preopen.

Remark 5.1. (a) By Remark 4.3(a), $(i, j)\mathrm{SO}(Y)$, $(i, j)\mathrm{PO}(Y)$, $(i, j)\alpha(Y)$ and $(i, j)\mathrm{SPO}(Y)$ are all m-structures on Y satisfying property $(\mathcal{B})$. Therefore, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-preopen) if and only if $f : (X, \tau_i) \to (Y, m_{ij})$ is m-open, where $m_{ij} = (i, j)\mathrm{SO}(Y)$ (resp. $(i, j)\mathrm{PO}(Y)$, $(i, j)\alpha(Y)$, $(i, j)\mathrm{SPO}(Y)$).

(b) A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-preopen if $F : (X, \tau_i) \rightarrow (Y, m_{ij})$ is m-open, where $m_{ij} = (i, j)\mathrm{PO}(Y)$.

Definition 5.3. Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y. A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-m-open* at $x \in X$ if $F : (X, \tau_i) \rightarrow (Y, m_{ij})$ is m-open at $x \in X$.

Remark 5.2. (a) A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is *(i, j)-m-open* at $x \in X$ if for each τ_i-open set U containing x, there exists $V \in m_{ij}$ containing $F(x)$ such that $V \subset F(U)$.

(b) By Remark 3.3, it follows from that a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-m-open, where $m_{ij} = m(\sigma_1, \sigma_2)$ has property $\mathcal{B}$, if and only if $F(U)$ is m-open for every τ_i-open set U of X.

(c) If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a function and $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a multifunction, then by Definiton 5.3 we obtain Definitions 5.1 and 5.2.

By Definition 5.3 and Theorems 3.1-3.5, we obtain the following theorems.

Theorem 5.1. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y with property $\mathcal{B}$. Then a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-m-open at $x \in X$ if and only if $x \in F^+(m_{ij}\text{-}\mathrm{Int}(F(U)))$ for every τ_i-open set U containing x.*

Theorem 5.2. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y with property $\mathcal{B}$. Then a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-m-open if and only if $F(U)$ is m_{ij}-open for every τ_i-open set U of X.*

Theorem 5.3. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y with property $\mathcal{B}$. For a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-m-open at $x \in X$;*

(2) *If $x \in i\mathrm{Int}(A)$ for $A \in \mathcal{P}(X)$, then $x \in F^+(m_{ij}\text{-}\mathrm{Int}(F(A))$;*

(3) *If $x \in i\mathrm{Int}(F^+(B))$ for $B \in \mathcal{P}(Y)$, then $x \in F^+(m_{ij}\text{-}\mathrm{Int}(B))$;*

(4) *If $x \in F^-(m_{ij}\text{-}\mathrm{Cl}(B))$ for $B \in \mathcal{P}(Y)$, then $x \in i\mathrm{Cl}(F^-(B))$.*

Theorem 5.4. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y with property $\mathcal{B}$. For a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is *(i, j)-m-open;*

(2) *$F(i\mathrm{Int}(A) \subset m_{ij}\text{-}\mathrm{Int}(F(A))$ for every subset A of X;*

(3) *$i\mathrm{Int}(F^+(B)) \subset F^+(m_{ij}\text{-}\mathrm{Int}(B))$ for every subset B of Y;*

(2) *$F(i\mathrm{Int}(A)) \subset m_{ij}\text{-}\mathrm{Int}(F(A))$ for every subset A of X;*

(3) *$i\mathrm{Int}(F^{+}(B)) \subset F^{+}(m_{ij}\text{-}\mathrm{Int}(B))$ for every subset B of Y;*

(4) *If $F^{-}(m_{ij}\text{-}\mathrm{Cl}(B)) \subset i\mathrm{Cl}(F^{-}(\mathrm{B}))$ for every subset B of Y.*

For a function $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we denote

$$D^0_{ij} = \{x \in X : F \text{ is not } (i, j)\text{-}m\text{-open at } x\},$$

then by Definition 5.3 and Theorem 3.5 we obtain the following theorem:

Theorem 5.5. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y with property $\mathscr{B}$, the following properties hold:*

$$\begin{aligned} D^0_{ij}(F) &= \cup_{U \in \mathcal{T}_i} \{U - F^{+}(m_{ij}\text{-}\mathrm{Int}(F(U)))\} \\ &= \cup_{A \in \mathcal{P}(X)} \{i\mathrm{Int}(A) - F^{+}(m_{ij}\text{-}\mathrm{Int}(F(A)))\} \\ &= \cup_{B \in \mathcal{P}(y)} \{i\mathrm{Int}(F^{+}(B)) - F^{+}(m_{ij}\text{-}\mathrm{Int}(B))\} \\ &= \cup_{B \in \mathcal{P}(y)} \{F^{-}(m_{ij}\text{-}\mathrm{Cl}(B)) - i\mathrm{Cl}(F^{-}(B))\}. \end{aligned}$$

6. NEW FORMS OF MODIFICATIONS OF OPEN MULTIFUNCTIONS

There are many modifications of open sets in topological spaces. In order to define some new modifications of open sets in a bitopological space, let recall θ-open sets and δ-open sets due to Velicko [33]. Let (X, τ) be a topological space. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of a subset A of X if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$). If $A = \mathrm{Cl}_\theta(A)$ (resp. $A = \mathrm{Cl}_\delta(A)$), then A is said to be *θ-closed* (resp. *δ-closed)* [33]. The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*). The union of all θ-open (resp. δ-open) sets contained in A is called the *θ-interior* (resp. *δ-interior*) of A and is denoted by $\mathrm{Int}_\theta(A)$ (resp. $\mathrm{Int}_\delta(A)$).

Definition 6.1. A subset A of a bitopological space (Y, σ_1, σ_2) is said to be

(1) *(i, j)-δ-semi-open* [27] if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-δ-preopen* [28] if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-δ-semi-preopen* (simply *(i, j)-δ-sp-open*) if there exists an (i, j)-δ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Definition 6.2. A subset A of a bitopological space (Y, σ_1, σ_2) is said to be

(1) *(i, j)-θ-semi-open* if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-θ-preopen* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

Let (Y, σ_1, σ_2) be a bitopological space. The family of (i, j)-δ-semi-open (resp. (i, j)-δ-preopen, (i, j)-δ-*sp*-open, (i, j)-θ-semi-open, (i, j)-θ-preopen, (i, j)-θ-*sp*-open) sets of (Y, σ_1, σ_2) is denoted by $(i, j)\delta SO(Y)$ (resp. $(i, j)\delta PO(Y)$, $(i, j)\delta SPO(Y)$, $(i, j)\theta SO(Y)$, $(i, j)\theta PO(Y)$, $(i, j)\theta SPO(Y)$).

Remark 6.1. Let (Y, σ_1, σ_2) be a bitopological space. The family $(i, j)\delta SO(Y)$, $(i, j)\delta PO(Y)$, $(i, j)\delta SPO(Y)$, $(i, j)\theta SO(Y)$, $(i, j)\theta PO(Y)$ and $(i, j)\theta SPO(Y)$ are all m-structures with property $\mathcal{B}$.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we can define many new types of (i, j)-m-open multifunctions. For example, in case $m_{ij} = (i, j)\delta SO(Y)$ (resp. $(i, j)\delta PO(Y)$, $(i, j)\delta SPO(Y)$, $(i, j)\theta SO(Y)$, $(i, j)\theta PO(Y)$, $(i, j)\theta SPO(Y)$) we can define new types of (i, j)-m-open multifunctions as follows:

Definition 6.3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-δ-semi-open* (resp. *(i, j)-δ-preopen*, *(i, j)-δ-sp-open*) if $F : (X, \tau_i) \to (Y, m_{ij})$ is (i, j)-m-open and $m_{ij} = (i, j)\delta SO(Y)$ (resp. $(i,j)\delta PO(Y)$, $(i, j)\delta SPO(Y)$).

Definition 6.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-θ-semi-open* (resp. *(i,j)-θ-preopen*, *(i, j)-θ-sp-open*) if $F : (X, \tau_i) \to (Y, m_{ij})$ is (i, j)-m-open and $m_{ij} = (i, j)\theta SO(Y)$ (resp. $(i, j)\theta PO(Y)$, $(i, j)\theta SPO(Y)$).

Conclusion. We can apply the characterizations established in Sections 5 to the multifunctions defined in Definitions 6.3 and 6.4 and also to multifunctions defined by using any m-structure $m_{ij} = m(\sigma_1, \sigma_2)$ with property $\mathcal{B}$ determined by σ_1 and σ_2 in a bitopological space (Y, σ_1, σ_2).

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ON PRECONTINUOUS AND α-PRECONTINUOUS MAPPINGS

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ABSTRACT : Precontinuity and α-precontinuity of mappings in topological spaces are considered. Another properties of these types of mappings and interrelationships with some other types are studied. Some observations concerning Hausdorffness, normality and β-connectedness of spaces are given.

Key words and phrases : Preopen, α-open, semi-open, simply open sets; α-precontinuity, precontinuity, semi-continuity, α-continuity, a.c.S.; submaximal, β-connected, Hausdorff, normal, 𝒟-spaces.
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1. INTRODUCTION

Mashhour et al. [28] introduced the notion of precontinuous mapping which coincides with almost continuity in the sense of Husain [16] (briefly: a.c.H.). Quite recently,Beceren and Noiri [4] have defined α-precontinuous mappings which are of stronger form of continuity than those precontinuous. Some interrelationships with other known classes of mappings can be found in [4]. In present paper we continue investigations concerning α-precontinuous and precontinuous mappings in context of several types of 'open' and 'continuous' mappings.

2. PRELIMINARIES

Throughout the paper, by (X, τ), (Y, σ), ... we mean topological spaces (briefly: spaces) on which no separation axioms are assumed unless explicitly stated. The Cartesian product topology for spaces (X, τ) and (Y, σ) will be denoted by $\tau \times \sigma$. Let S be a subset of an (X, τ). The closure of S and the interior of S (both in (X, τ)) are denoted by $\mathrm{cl}_\tau(S)$ (or $\mathrm{cl}(S)$) and $\mathrm{int}_\tau(S)$ (or $\mathrm{int}\,(S)$), respectively. A subset $S \subset X$ is said to be ***regular open*** (resp. ***regular closed***) in (X, τ) if $S = \mathrm{int}\,(\mathrm{cl}(S))$ (resp. $S = \mathrm{cl}(\mathrm{int}(S))$). The family of all closed (resp. regular open, regular closed) subsets of a space (X, τ) will be denoted by $\mathrm{c}(X, \tau)$ (resp. $\mathrm{RO}\,(X, \tau)$, $\mathrm{RC}(X, \tau)$). A subset S of (X, τ) is said to be ***α-open*** [32] (resp. ***semi-open*** [21], ***preopen*** [28], ***semi-preopen*** [2] or equivalently ***β-open*** [1]) if $S \subset \mathrm{int}\,(\mathrm{cl}(\mathrm{int}(S)))$ (resp. $S \subset \mathrm{cl}(\mathrm{int}(S))$, $S \subset \mathrm{int}(\mathrm{cl}(S))$, $S \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$). The complement of an α-open (resp. semi-open, preopen, semi-preopen) set is called ***α-closed*** (resp. ***semi-closed, preclosed, semi-preclosed***). The family of all α-open (resp. semi-open, preopen, semi-preopen) subsets of (X, τ) will be denoted by

τ^α (resp. $SO(X, \tau)$, $PO(X, \tau)$, $SPO(X, \tau)$). The family of all semi-closed (resp. preclosed, semi-preclosed) subsets of (X, τ) we denote by $SC(X, \tau)$ (resp. $PC(X, \tau)$, $SPC(X, \tau)$). The following incusions are known for any space (X, τ) : $\tau \subset \tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$ ([39, Lemma 3.1] or [45, Lemma 2]), $SO(X, \tau) \subset SPO(X, \tau)$ and $PO(X, \tau) \subset SPO(X, \tau)$. For each (X, τ) the family τ^α forms a topology on X [32, Proposition 2] which is strictly distinct from τ, in general. We have $(\tau^\alpha)^\alpha = \tau^\alpha$ [32, Proposition 10] for each (X, τ). If $\mathcal{A} \subset SO(X, \tau)$ (resp. $\mathcal{A} \subset PO(X, \tau)$), then $\cup \mathcal{A} \in SO(X, \tau)$ [21, Theorem 2] (resp. $\cup \mathcal{A} \in PO(X, \tau)$ [28]).

With a standard method, for any space (X, τ) and a subset $S \subset X$ one defines [2] the ***preinterior*** of S ($\mathrm{pint}_\tau(S)$), the ***semi-preinterior*** of S ($\mathrm{spint}_\tau(S)$), the ***preclosure*** of S ($\mathrm{pcl}_\tau(S)$), and the ***semi-preclosure*** of S ($\mathrm{spcl}_\tau(S)$) as respectively: the union of all preopen (resp. semi-preopen) subsets of (X, τ) contained in S and the intersection of all preclosed (resp. semi-preclosed) subsets of (X, τ) containing S.

The following formulas are known: $\mathrm{pcl}_\tau(S) = S \cup \mathrm{cl}_\tau(\mathrm{int}_\tau(S))$ [2, Theorem 1.5(e)], $\mathrm{pint}_\tau(S) = S \cap \mathrm{int}_\tau(\mathrm{cl}_\tau(S))$ [2, Theorem 1.5(f)]. A mapping $f: (X, \tau) \to (Y, \sigma)$ is said to be *α-**precontinuous*** [4] (resp. ***precontinuous*** [28], ***semicontinuous*** [21], *α-**continuous*** [30], *α-**irresolute*** [27], ***almost continuous in the sense of Singal and Singal*** [48] or briefly ***a.c.S.***) if $f^{-1}(V) \in PO(X, \tau)$ (resp. $f^{-1}(V) \in PO(X, \tau)$, $f^{-1}(V) \in SO(X, \tau)$, $f^{-1}(V) \in \tau^\alpha$, $f^{-1}(V) \in \tau^\alpha$, $f^{-1}(V) \in \tau$) for every set $V \in \sigma^\alpha$ (resp. $V \in \sigma$, $V \in \sigma$, $V \in \sigma$, $V \in \sigma^\alpha$, $V \in RO(X, \tau)$ [48, Theorem 2.2(b)]).

A mapping $f : (X, \tau) \to (Y, \sigma)$ is called ***contra-continuous*** [7] (resp. ***contra-semi-continuous*** [8], ***contra-precontinuous*** [17]) if $f^{-1}(V) \in c(X, \tau)$ (resp. $f^{-1}(V) \in SC(X, \tau)$, $f^{-1}(V) \in PC(X, \tau)$) for every $V \in \sigma$.

3. α-PRECONTINUOUS MAPPINGS

In [4] the authors gave several characterizations of α-precontinuity. In the following theorem we offer another characterizations of this type of continuity.

Theorem 1. *For a mapping* $f : (X, \tau) \to (Y, \sigma)$ *the following statements are eqivalent:*

(1) f is α-precontinuous.

(2) $\mathrm{pcl}_\tau(f^{-1}(B)) \subset f^{-1}(cl_{\sigma^\alpha}(B))$ *for every subset* $B \subset Y$.

(3) $f(\mathrm{pcl}_\tau(A)) \subset cl_{\sigma^\alpha}(f(A))$ *for every subset* $A \subset X$.

(4) $f^{-1}(\mathrm{int}_{\sigma^\alpha}(S)) \in \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(S)))$ *for every subset* $S \subset Y$.

(5) $f^{-1}(\mathrm{int}_{\sigma^\alpha}(S)) \subset \mathrm{pint}_\tau(f^{-1}(S))$ *for every subset* $S \subset Y$.

Proof. (1) $\Rightarrow$ (2). Let f be α-precontinuous. Then $\mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(B))) \subset f^{-1}(\mathrm{cl}_{\sigma^\alpha}(B))$ for every subset $B \subset Y$ [4, Theorem 3.1(f)]. Our inclusion follows from [2, Theorem 1.5(e)].

(2) $\Rightarrow$ (1). Obvious by [2, Theorem 1.5(e)] and [4, Theorem 3.1(f)].

(1) $\Leftrightarrow$ (3). This follows immediately from [4, Theorem 3.1(g)] and [2, Theorem 1.5(e)].

(1) $\Leftrightarrow$ (4). Let S be any subset of Y and let $B = Y \setminus S$. Utilizing [4, Theorem 3.1(f)] we calculate as follows: $\mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(B))) \subset f^{-1}(\mathrm{cl}_{\sigma^\alpha}(B))$ iff $X \setminus f^{-1}(\mathrm{cl}_{\sigma^\alpha}(B)) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(X \setminus f^{-1}(B)))$ iff $f^{-1}(\mathrm{int}_{\sigma^\alpha}(S) \in \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(S)))$.

(4) $\Leftrightarrow$ (5). Apply [2, Theorem 1.5(f)].

The following result has been obtained by Beceren and Noiri.

Theorem 2. [4, Theorem 3.2]. *Let (X, τ) and (Y, σ) be arbitrarily chosen spaces and let the graph mapping $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ for an $f : (X, \tau) \to (Y, \sigma)$ be given via $g(x) = (x, f(x))$ for each $x \in X$. If g is α-precontinuous then f is α-precontinuous.*

For a.c.S., a.c.H., and w.c. [35] mappings, theorems of the above type are reversible; see respectively [23, Theorems 1&2], [35, Theorem 1]. Also, in the cases of a.α.c. [41], s.w.c. [42], p.a.α.c. [11], and p.s.w.c. [11] mappings, reversibilities of this kind are possible; see [11].

We are to shew that under a certain condition imposed on (X, τ) and (Y, σ), the converse of Theorem 2 holds.

We offer another proof of the following assertion.

Lemma 1. [13, p. 136]. *Let (X, τ) be a space, $U \in \tau^\alpha$ and $V \in$ PO(X, τ). Then $U \cap V \in$ PO(X, τ).*

Proof. We have $U \cap V \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(U))) \cap \mathrm{int}(\mathrm{cl}(V)) \subset \mathrm{int}(\mathrm{cl}(U)) \cap \mathrm{int}(\mathrm{cl}(V))$. From [39, Lemma 3.5] we infer that $U \cap V \in$ PO(X, τ).

Definition 2. *Let (X, τ) and (Y, σ) be spaces. Then*

$$(\tau \times \sigma)^\alpha = (\tau^\alpha \times \sigma^\alpha)^\alpha$$

Proof. First, we will show the inclusion

(1) $\tau^\alpha \times \sigma^\alpha \subset (\tau \times \sigma)^\alpha$.

Let $S \in \tau^\alpha \times \sigma^\alpha$. Then $S = \cup_{i \in J} S_i^1 \times S_i^2$ where $S_i^1 \in \tau^\alpha$ and $S_i^1 \in \sigma^\alpha$ for each $i \in J$. So,

$$S \subset \bigcup_{i \in J}\Big(\mathrm{int}_\tau\big(\mathrm{cl}_\tau\big(\mathrm{int}_\tau(S_i^1)\big)\big) \times \mathrm{int}_\sigma\big(\mathrm{cl}_\sigma\big(\mathrm{int}_\sigma(S_i^2)\big)\big)\Big)$$

$$= \bigcup_{i \in J}\Big(\mathrm{int}_{\tau\times\sigma}\big(\mathrm{cl}_{\tau\times\sigma}\big(\mathrm{int}_{\tau\times\sigma}(S_i^1 \times S_i^2)\big)\big)\Big) \subset \mathrm{int}_{\tau\times\sigma}\Big(\mathrm{cl}_{\tau\times\sigma}\big(\mathrm{int}_{\tau\times\sigma}(S)\big)\Big)$$

Therefore, $S \in (\tau \times \sigma)^\alpha$. The next inclusion to be shown is

(2) $(\tau \times \sigma)^\alpha \subset (\tau^\alpha \times \sigma^\alpha)^\alpha$.

Let $W \in (\tau \times \sigma)^\alpha$. Then we have

$$W \subset \operatorname{int}_{\tau\times\sigma}\left(cl_{\tau\times\sigma}\left(\operatorname{int}_{\tau\times\sigma}(W)\right)\right) \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(cl_{\tau\times\sigma}\left(\operatorname{int}_{\tau\times\sigma}(W)\right)\right).$$

By [14, Lemma 1(i)] and (1) we get

$$W \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{cl}_{(\tau\times\sigma)^\alpha}\left(\operatorname{int}_{\tau\times\sigma}(W)\right)\right) \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{cl}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{int}_{\tau^\alpha\times\sigma^\alpha}(W)\right)\right).$$

Thus, $W \in \left(\tau^\alpha \times \sigma^\alpha\right)^\alpha$. We shall show now that

(3) $\left(\tau^\alpha \times \sigma^\alpha\right)^\alpha \subset (\tau \times \sigma)^\alpha$.

Let $V \in \left(\tau^\alpha \times \sigma^\alpha\right)^\alpha$. Hence $V \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{cl}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{int}_{\tau^\alpha\times\sigma^\alpha}(V)\right)\right)$. By (1) and [14, Lemma 1(i)] we obtain

$$V \subset \operatorname{int}_{(\tau\times\sigma)^\alpha}\left(\operatorname{cl}_{(\tau^\alpha\times\sigma^\alpha)^\alpha}\left(\operatorname{int}_{\tau^\alpha\times\sigma^\alpha}(V)\right)\right).$$

By (2) and (1) we get $V \subset \operatorname{int}_{(\tau\times\sigma)^\alpha}\left(\operatorname{cl}_{(\tau\times\sigma)^\alpha}\left(\operatorname{int}_{(\tau\times\sigma)^\alpha}(V)\right)\right)$. This shows that

$$V \in \left((\tau \times \sigma)^\alpha\right)^\alpha = (\tau \times \sigma)^\alpha;$$

see [32]. Eventually, inclusions (2) and (3) complete the proof.

Corollary 1. *Let* (X, τ) *and* (Y, σ) *be such spaces that*

(4) $\left(\tau^\alpha \times \sigma^\alpha\right)^\alpha \subset \tau^\alpha \times \sigma^\alpha$

Then $(\tau \times \sigma)^\alpha = \tau^\alpha \times \sigma^\alpha$.

Theorem 2. *Let* (X, τ), (Y, σ) *be spaces and let mappings* f *and* g *be as in Theorem* 2. *If* (4) *holds and if* f *is* α*-precontinuous, then* g *is* α*-precontinuous.*

Proof. Suppose f is α-precontinuous. Let $x \in X$ and let $W \in (\tau \times \sigma)^\alpha$ be any set containing $g(x)$. By (4) and by Corollary 1 there exist sets $U \in \tau^\alpha$, $V \in \sigma^\alpha$, such that $g(x) = (x, f(x)) \in U \times V \subset W$. Since f is α-precontinuous, there is a set $U_1 \in \mathrm{PO}(X, \tau)$ containing x such that $f(U_1) \subset V$ [4, Theorem 3.1(c)]. Thus, $f(U \cap U_1) \subset V$ where $x \in U \cap U_1 \in \mathrm{PO}(X, \tau)$ by Lemma 1. So we obtain $g(U \cap U_1) \subset U \times V \subset W$. This shows that g is α-precontinuous.

For every ***totally disconnected*** space (X, τ) (each open set is closed), we have $\tau = \tau^\alpha$ [19, Theorem 3.3].

Corollary 2. *Let (X, τ) and (Y, σ) be such spaces that the space $(X \times Y, \tau \times \sigma)$ is totally disconnected and let $f : (X, \tau) \to (Y, \sigma)$. Then, the graph mapping g of f is α-precontinuous if and only if f is α-precontinuous.*

Theorem 3. *Let for $i = 1, 2$, (X_i, τ_i) be arbitrary spaces and (Y_i, σ_i) be spaces fulfilling the condition $\left(\sigma_1^\alpha \times \sigma_2^\alpha\right)^\alpha \subset \sigma_1^\alpha \times \sigma_2^\alpha$. Then, mappings $f_i : (X_i, \tau_i) \to (Y_i, \sigma_i)$, $i = 1, 2$, are α-precontinuopus if and only if the product mapping $f : (X_1 \times X_2, \tau_1 \times \tau_2) \to (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$, defined via $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$, is α-precontinuous.*

Proof. Sufficiency. [4, Theorem 3.4].

Necessity. Let f_i be α-precontinuous, $i = 1, 2$, and let $W \in (\sigma_1 \times \sigma_2)^\alpha$ By Corollary 1 we have $W \in \sigma_1^\alpha \times \sigma_2^\alpha$, and so $W = \cup_{j \in \mathcal{J}} W_1^j \times W_2^j$ where $W_1^j \in \sigma_1^\alpha$, $W_2^j \in \sigma_2^\alpha$ for each $j \in \mathcal{J}$. Using [4, Theorem 3.1(d)], for any chosen $j \in \mathcal{J}$ we calculate as follows:

$$f^{-1}\left(W_1^j \times W_2^j\right) = f_1^{-1}\left(W_1^j\right) \times f_2^{-1}\left(W_2^j\right) \subset \mathrm{int}_{\tau_1}\left(\mathrm{cl}_{\tau_1}\left(f_1^{-1}\left(W_1^j\right)\right)\right) \times \mathrm{int}_{\tau_2}\left(\mathrm{cl}_{\tau_2}\left(f_2^{-1}\left(W_2^j\right)\right)\right)$$

$$= \mathrm{int}_{\tau_1 \times \tau_2}\left(\mathrm{cl}_{\tau_1 \times \tau_2}\left(f^{-1}\left(W_1^j \times W_2^j\right)\right)\right) \subset \mathrm{int}_{\tau_1 \times \tau_2}\left(\mathrm{cl}_{\tau_1 \times \tau_2}\left(f^{-1}(W)\right)\right).$$

Therefore, $f^{-1}(W) \subset \mathrm{int}_{\tau_1 \times \tau_2}\left(\mathrm{cl}_{\tau_1 \times \tau_2}\left(f^{-1}(W)\right)\right)$ and consequently f is α-precontinuous.

A subset S of a space (X, τ) is said to be ***simply open*** [6], if $S = O \cup N$ where $O \in \tau$ and N is nowhere dense in (X, τ). A mapping $f : (X, \tau) \to (Y, \sigma)$ is called ***simply continuous*** [6] if the preimage $f^{-1}(G)$ is simply open in (X, τ) for each $G \in \sigma$.

Each semi-open subset of a space (X, τ) is simply open in (X, τ) [21, Theorem 7]. Thus, any semi-continuous mapping is simply continuous. The converse is not the truth [6, Example 1.1.2].

Lemma 3. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is* α-precontinuous and contra-continuous, then $f^{-1}(S) \in \mathrm{PC}(X, \tau)$ *for every simply open subset S of (Y, σ).*

Proof. Let $S = O \cup N$ be simply open in (Y, σ) ($O \in \tau$ and N is nowhere dense). Then, $f^{-1}(S) = f^{-1}(O) \cup f^{-1}(N)$ where $U_1 = f^{-1}(O) \in \mathrm{c}(X, \tau)$ and $U_2 = f^{-1}(N) \in \mathrm{PC}(X, \tau)$ [4, Theorem

3.7]. Applying a dual equality to that of [39, emma 3.5] we obtain $f^{-1}(S) = U_1 \cup U_2 \supset \mathrm{cl}_\tau(\mathrm{int}_\tau(U_1)) \cup \mathrm{cl}_\tau(\mathrm{int}_\tau(U_2) = \mathrm{cl}_\tau(\mathrm{int}_\tau(U_1 \cup U_2)) = \mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(S)))$. Therefore $f^{-1}(S) \in \mathrm{PC}(X, \tau)$.

Theorem 4. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \nu)$ be given mappings. If g is simply continuous, f is α-precontinuous and contra-continuous, then $g \circ f : (X, \tau) \to (Z, \nu)$ is contra-precontinuous.*

Proof. It follows from Lemma 3.

Theorem 5. *Let a mapping $f : (X, \tau) \to (Y, \sigma)$ be α-precontinuous, continuous, contra-semicontinuous, and let (Y, σ) be a $\mathcal{T}_1$-space. Then, for each $y \in Y$ and for each $V \in \sigma$ such that $y \in \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$ we have $f^{-1}(V \cup \{y\}) \in \mathrm{RO}(X, \tau)$.*

Proof. Let $V \in \sigma$ be arbitrarily chosen. Remark that if $y \in V$, then up to continuity and contra-semicontinuity of f we get $f^{-1}(V) \in \mathrm{RO}(X, \tau)$. By α-precontinuity of f [4, Theorem 3.8] for every $y \in \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$ we have $f^{-1}(V \cup \{y\}) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V \cup \{y\}))) = \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V)) \cup \mathrm{cl}_\tau(f^{-1}(\{y\}))) \subset \mathrm{cl}_\tau(f^{-1}(\{y\})) \cup \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V)))$ [2, Lemma 1.1.(b)]. Since f is continuous and (Y, σ) is a $\mathcal{T}_1$-space, $\mathrm{cl}_\tau(f^{-1}(\{y\})) = f^{-1}(\{y\})$. From contra-semicontinuity of f we infer that $f^{-1}(\{y\}) \cup \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V))) \subset f^{-1}(V \cup \{y\})$. Hence clearly, $f^{-1}(V \cup \{y\}) \in \mathrm{RO}(X, \tau)$.

Remark 1. Continuity and α-precontinuiuty are independent of each other [4, Examples 2.1 & 2.2]

Theorem 6. *Assume an $f : (X, \tau) \to (Y, \sigma)$ has the following : for each $y \in Y$ and each $V \in \sigma$ with $y \in \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$, the preimage $f^{-1}(V \cup \{y\})$ is in $\mathrm{RO}(X, \tau)$. Then, f is α-precontinuous, continuous, and contra-semicontinuous.*

Proof. The α-precontinuity of f is clear by [4, Theorem 3.8]. Let $V \in \sigma$ be arbitrary and let $y \in V$. By $f^{-1}(V) \in \mathrm{RO}(X, \tau)$ it follows evidently that f is continuous and contra-semicontinuous.

4. PRECONTINUOUS MAPPINGS

It is known that each α-precontinuous mapping is precontinuous and that the converse is not true, in general [4, Remark]. Thus, results concerning precontinuous mappings hold also in the α-precontinuity case. Some of them are not mentioned in [4] the reader is advised to see for instance [29, Theorems 2.3 & 2.4 & 2.5].

We recall that for every two a.c.S. mappings f_1, f_2 from a space (X, τ) into a Hausdorff space (Y, σ), the set $\{x \in X : f_1(x) = f_2(x)\}$ is closed in (X, τ) [24, Theorem 4]. This result is obviously true for any two mappings f_1, f_2 with a stronger type of continuity than a.c.S. (see for instance [40, Diagram p. 249]), in particular for ***completely continuous*** mappings [3]. On the other hand, the following is evident.

Remark 2. A mapping $f : (X, \tau) \to (Y, \sigma)$ is precontinuous and contra-semicontinuous if and only if it is completely continuous.

Theorem 7. *Let* (X, τ) *be arbitrary,* (Y, σ) *be Hausdorff, and let mappings* $f_1, f_2 : (X, \tau) \to (Y, \sigma)$ *be given. If* f_1 *is* α*-continuous and* f_2 *is precontinuous, then* $A = \{x \in X : f_1(x) = f_2(x)\} \in \mathrm{PC}(X, \tau)$.

Proof. Let x be any point of $X \setminus A$. Hence $f_1(x) \neq f_2(x)$ and since (Y, σ) is Hausdorff, there exist sets $V_1, V_2 \in \sigma$ such that $f_1(x) \in V_1$, $f_2(x) \in V_2$, and $V_1 \cap V_2 = \emptyset$.

By Lemma 1 we have $x \in U_x = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \in \mathrm{PO}(X, \tau)$. We will show that $U_x \subset X \setminus A$. Suppose not. There exists a point $x' \in U_x$ such that $x' \in A$. Hence $f_1(x') = f_2(x') \in V_2$. This implies that $x' \in f_1^{-1}(V_1) \cap f_1^{-1}(V_2) = \emptyset$, a contradiction. Thus, the set $X \setminus A$ is preopen and consequently $A \in \mathrm{PC}(X, \tau)$.

Remark 3. (a) [25, Example 1] and [48, Example 2.1] show that precontinuity and a.c.S. are independent notions [23, p. 413].

(b) [39, Examples 3.9 & 3.10] show that α-continuity and a.c.S. are independent of each other.

An analysis of the proof of Theorem 7 leads to the following slight improvement of [15, Theorem 2.6(1)]. We use the fact that in any (X, τ), if $U \in \tau^{\alpha}$ and $V \in \mathrm{SO}(X, \tau)$, then $U \cap V \in \mathrm{SO}(X, \tau)$ [32].

Theorem 8. *Let* (X, τ) *be arbitary,* (Y, σ) *be Hausdorff, and* $f_1, f_2 : (X, \tau) \to (Y, \sigma)$. *If* f_1 *is* α*-continuous and* f_2 *is semi-continuous, then the set* $A = \{x \in X : f_1(x) = f_2(x)\} \in \mathrm{SC}(X, \tau)$.

Remark 4. (a) [21, Example 8] and [38, Example 4.1] show that semi-continuity and a.c.S. are independent of each other.

(b) [37, Examples 2.3] shows that there exists an α-continuous mapping which is not continuous. Obviously, each continuous map is α-continuous.

(c) [31, Examples 3.1 & 3.2] show that semi-continuity and precontinuity are independent of each other.

If mappings $f_1, f_2 : (X, \tau) \to (Y, \sigma)$ are both α-continuous (hence precontinuous), but the space (Y, σ) is not Hausdorff, then the set A from Theorems 7 and 8 mut not be even semi-preclosed in (X, τ). It is worth to see also [39, Theorem 4.9].

Example 1. *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}\}$, *and* $\sigma = \{\emptyset, Y, \{a\}\}$. *Define*

$f_1, f_2 : (X, \tau) \to (Y, \sigma)$ *as follows:* $f_1(c) = b$, $f_2(a) = c$, $f_1(b) = f_2(b) = f_1(c) = f_2(c) = a$. *Then* f_1 *and* f_2 *are* α*-continuous and the set* $\{x \in X : f_1(x) = f_2(x)\} = \{b, c\} \notin \mathrm{SPC}(X, \tau)$.

Using Theorem 8 and [15, Theorem 2.4] we slightly improve [15, Theorem 2.6(2)].

Corollary 3. *Let* (X, τ) *be arbitrary,* (Y, σ) *be Hausdorff, and* $f_1, f_2 : (X, \tau) \to (Y, \sigma)$. *If* f_1 is α*-continuous,* f_2 *is semi-continuous, and* $f_1 = f_2$ *on a dense subset of* (X, τ), *then* $f_1 = f_2$ *on* X.

Definition 1. *A subset* S *of a space* (X, τ) *is said to be* **p-dense** *in* (X, τ) *if* $\mathrm{pcl}_\tau(S) = X$.

It is obvious that every subset p-dense in (X, τ) is dense in (X, τ), but the converse is not always true.

Example 2. *Consider the space* **R** *of all reals with Euclidean topology* τ_e, $S =$ **Q** (**Q** *the set of all rationals*). *By* [2, Theorem 1.5(e)] *we have* $\mathrm{pcl}_{\tau_e}(S) = S$.

The next result follows immediately from Theorem 7.

Corollary 4. *Let* (X, τ) *be arbitrary,* (Y, σ) *be Hausdorff, and* $f_1, f_2 : (X, \tau) \to (Y, \sigma)$. *If* f_1 *is* α*-continuous,* f_2 *is precontinuous, and* $f_1 = f_2$ *on a* **p**-*dense subset of* (X, τ), *then* $f_1 = f_2$ *on* X.

Theorem 9. *Let* (X, τ) *be arbitrary and* $S \in \mathrm{SO}(X, \tau)$. *Then* S *is* p*-dense in* (X, τ) *if and only if* S *is dense in* (X, τ).

Proof. Sufficiency. Apply [2, Theorem 1.5(e)] and [34, Lemma 2].

Theorem 10. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a precontinuous surjection. If a set* $S \in \mathrm{SO}(X, \tau)$ *is dense in* (X, τ), *then* $f(S)$ *is dense in* (Y, σ).

Proof. [20, Proposition 3.1]. The reader is advised to compare the characterization (3) of precontinuous mapings given in [47, Theorem 6].

Theorem 11. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an* α*-precontinuous surjection. If a set* $S \subset X$ *is* p*-dense in* (X, τ), *then* $f(S)$ *is dense in* (Y, σ).

Proof. This follows from Theorem 1(3).

A pace (X, τ) is called a ***𝒟-space*** [22] if each nonempty set $V \in \tau$ is dense in (X, τ). By [22, Theorem 1] and [49, Theorem 17], being a 𝒟-space, ***S-connectedness*** (X connot be expressed as the union of two nonempty disjoint semi-open subsets), and ***irreducibility*** are equivalent.

A space (X, τ) is said to be ***submaximal*** if for every dense subset S of (X, τ) we have $S \in \tau$.

Theorem 12. *Let* (X, τ) *be a* $\mathcal{D}$*-space and* (Y, σ) *be submaximal. If a surjection* $f : (X, \tau) \to (Y, \sigma)$ *is precontinuous then* f *is open.*

Proof. [20, Proposition 3.1] or [47, Theorem 6].

Mashour et al. [28, theorem 1] established the following characterization of precontinuous mappings: $f : (X, \tau) \to (Y, \sigma)$ is precontinuous if and only if $f(\mathrm{cl}_\tau(\mathrm{int}_\tau(U))) \subset \mathrm{cl}_\sigma(f(U))$ for every $U \subset X$. By this and by [2, Theorem 1.5(e)] one easily obtains the following.

Lemma 4. *Let* (X, τ) *and* (Y, σ) *be any spaces. A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is precontinuous if and only if* $f(\mathrm{pcl}_\tau(U) \subset \mathrm{cl}_\sigma(f(U))$ *for every* $U \subset X$.

Theorem 13. *Let* $f : (X_i, \tau_i) \to (Y_i, \sigma_i)$, $i = 1, 2$, *be* (X, τ) *be precontinuous surjections. If a set* $S_1 \times S_2 \subset X_1 \times X_2$ *is p-dense in* $(X_1 \times X_2, \tau_1 \times \tau_2)$ *then the product's image* $f(S_1 \times S_2)$ *is dense in* $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$.

Proof. Let a set $S_1 \times S_2 \subset X_1 \times X_2$ be p-dense in $X_1 \times X_2$. Utilizing [12, Lemma 5.2] and Lemma 4 we get $Y_1 \times Y_2 = f(\mathrm{pcl}(S_1 \times S_2)) \subset f\left(\mathrm{pcl}_{\tau_1}(S_1) \times \mathrm{pcl}_{\tau_2}(S_2)\right) = f_1\left(\mathrm{pcl}_{\tau_1}(S_1)\right)$

$$\times f_2\left(\mathrm{pcl}_{\tau_2}(S_2)\right) \subset \mathrm{cl}_{\sigma_1}\left(f_1(S_1)\right) \times \mathrm{cl}_{\sigma_2}\left(f_2(S_2)\right) = \mathrm{cl}_{\sigma_1 \times \sigma_2}\left(f(S_1 \times S_2)\right).$$

Theorem 14. *Let* (X_i, τ_i) *be* $\mathcal{D}$*-spaces and* (Y_i, σ_i) *be submaximal,* $i = 1, 2$. *If mappings* $f : (X_i, \tau_i) \to (Y_i, \sigma_i)$ *are precontinuous surjections then the product mapping* f *is open.*

Proof. Without difficulties we infer from [28, Theorem 1] that $f(U_1 \times U_2) \in \sigma_1 \times \sigma_2$ for any $U_1 \in \tau$, $U_2 \in \tau$; a calculation is similar to that in the proof of Theorem 13.

A space (X, τ) is said to be β-***connected*** [44] if X cannot be expressed as the union of two nonempty disjoint semi-preopen subsets of (X, τ). A space (X, τ) is β-disconnected if it is not β-connected.

Definiton 2. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* $\mathcal{P}$***-open*** *if* $f(V) \in \sigma$ *for each* $V \in \mathrm{PO}(X, \tau)$.

Recall that Jankovic [20] calls a mapping p-***open*** if it preserves preopen sets. Obviously each $\mathcal{P}$-open mapping is open, but these concepts are strictly distinct.

Example 3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a, b, c\}, \{a\}, \{b, c\}\}$. *The identity mapping* $\mathrm{id} : (X, \tau) \to (X, \tau)$ *is open but not* $\mathcal{P}$*-open, because* $\mathrm{id}(\{b\}) = \{b\} \notin \tau$.

Theorem 15. *Let* (X, τ) *be* β*-connected and* (Y, σ) *be submaximal. If* $f : (X, \tau) \to (Y, \sigma)$ *is a precontinuous surjection then it is* $\mathcal{P}$*-open.*

Proof. We use [18, Theorem 3.1(2)] and Lemma 4.

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We complete characterizations of β-connected spaces given in [18, Theorem 3.1] (see also [43]) with the following.

Theorem 16. *For any space* (X, τ) *the following are equivalent:*

(1) (X, τ) *is* β*-connected;*

(2) $\operatorname{pint}(\operatorname{pcl}(V)) = X$ *for each nonempty* $V \in \mathrm{PO}(X, \tau)$;

(3) $\operatorname{spint}(\operatorname{pcl}(V)) = X$ *for each nonempty* $V \in \mathrm{SPO}(X, \tau)$;

(4) $\operatorname{pint}(\operatorname{spcl}(V)) = X$ *for each nonempty* $V \in \mathrm{PO}(X, \tau)$;

(5) $\operatorname{spint}(\operatorname{spcl}(V)) = X$ *for each nonempty* $V \in \mathrm{SPO}(X, \tau)$.

Proof. Use respective parts of [18, Theorem 3.1]

Lemma 5. *In every topological space* (X, τ) *and for any* $W \subset X$ *we have*

(a) $\operatorname{pcl}_\tau(\operatorname{pint}_\tau(W)) \in \mathrm{SPO}(X, \tau)$,

(b) $\operatorname{pcl}_\tau(\operatorname{spint}_\tau(W)) \in \mathrm{SPO}(X, \tau)$,

(c) $\operatorname{spcl}_\tau(\operatorname{pint}_\tau(W)) \in \mathrm{SPO}(X, \tau)$.

Proof. (a) By [2, Theorem 1.5(e)] we have what follows.

$$\begin{aligned}
\operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pcl}(\operatorname{pint}(W))))) &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W)))))) \\
&= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W)))) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W))) \\
&\supset \operatorname{pint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W))) = \operatorname{pcl}(\operatorname{pint}(W)),
\end{aligned}$$

because $\operatorname{pint}(W) \in \mathrm{PO}(X, \tau)$.

(b) By [2, Theorem 1.5(e)] we get

$$\begin{aligned}
\operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pcl}(\operatorname{spint}(W))))) &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{spint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{spint}(W)))))) \\
&= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{spint}(W)))) \cup \operatorname{cl}(\operatorname{int}(\operatorname{spint}(W))) \\
&\supset \operatorname{spint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{spint}(W))) = \operatorname{pcl}(\operatorname{spint}(W)).
\end{aligned}$$

(c) Applying [2, Theorem 2.15] we obtain

$$\begin{aligned}
\operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{spcl}(\operatorname{pint}(W))))) &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{pint}(W))))))) \\
&= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W)))) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W)))) \\
&\supset \operatorname{pint}(W) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{pint}(W)))) = \operatorname{spcl}(\operatorname{pint}(W)).
\end{aligned}$$

Theorem 17. *Let* (X, τ) *be* β*-disconnected. Then, the space* X *allows the following partitions* [9, p.13]:

(a) $\{\mathrm{pint}(\mathrm{pcl}(S)), \mathrm{pcl}(\mathrm{pint}(X \setminus S))\} \subset \mathrm{SPO}(X, \tau)$ *for a certain nonempty* $S \in \mathrm{PO}(X, \tau)$,

(b) $\{\mathrm{pint}(\mathrm{spcl}(S)), \mathrm{pcl}(\mathrm{spint}(X \setminus S))\} \subset \mathrm{SPO}(X, \tau)$ *for a certain nonempty* $S \in \mathrm{PO}(X, \tau)$,

(c) $\{\mathrm{spint}(\mathrm{pcl}(S)), \mathrm{spcl}(\mathrm{pint}(X \setminus S))\} \subset \mathrm{SPO}(X, \tau)$ *for a certain nonempty* $S \in \mathrm{SPO}(X, \tau)$,

(d) $\{\mathrm{spint}(\mathrm{spcl}(S)), \mathrm{spcl}(\mathrm{spint}(X \setminus S))\} \subset \mathrm{SPO}(X, \tau)$ *for a certain nonempty* $S \in \mathrm{SPO}(X, \tau)$.

Proof. (a) From β-disconnectedness of (X, τ) and from Theorem 16(2') we infer that there exists a nonempty $S \in \mathrm{PO}(X, \tau)$ such that $U_1 = \mathrm{pint}(\mathrm{pcl}(S)) \neq X$. Obviously $U_1 \in \mathrm{SPO}(X, \tau)$. We shall show that $U_1 \neq \emptyset$. Suppose not. By [2, Theorem 1.5(f)] We get $\emptyset = \mathrm{pcl}(S) \cap \mathrm{int}(\mathrm{cl}(\mathrm{pcl}(S)))$. Applying [39, Lemma 3.5] Lemma 3.5] we have $\emptyset = \mathrm{int}(\mathrm{cl}(\mathrm{pcl}(S))) \supset S$ and so $S = \emptyset$. A contradiction. Put now $U_2 = X \setminus \mathrm{pint}(\mathrm{pcl}(S)) = \mathrm{pcl}(\mathrm{pint}(X \setminus S))$. Clearly, $X \neq U_2 \neq \emptyset$ and by Lemma 5(a), $U_2 \in \mathrm{SPO}(X, \tau)$.

Proofs for (b) – (d) are similar to the above. We use respective parts of Theorem 16, Lemma 5, and [18, Lemma 3.1].

(b) We shall show only that $U_1 = \mathrm{pint}(\mathrm{spcl}(S)) \neq \emptyset$, where a nonempty $S \in \mathrm{PO}(X, \tau)$ is such that $U_1 \neq X$. Suppose not. By [2, Theorem 1.5(f)] and [39, Lemma 3.5] we obtain $\emptyset = \mathrm{spcl}\ (S) \cap \mathrm{int}(\mathrm{cl}(\mathrm{spcl}(S))) = \mathrm{int}(\mathrm{cl}(\mathrm{spcl}(S)))$. So, $\emptyset = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{spcl}(S)))) \supset \mathrm{spcl}\ (S)$ [18, Lemma 3.1(2)], a contradiction.

(c) We shall show only that $U_1 = \mathrm{spint}\ (\mathrm{pcl}(S)) \neq \emptyset$, where a nonempty $S \in \mathrm{SPO}(X, \tau)$ is such that $U_1 \neq X$. Suppose not. By [2, Theorem 3.21(a)] we have $\emptyset = \mathrm{spint}\ (\mathrm{pcl}(S)) \supset \mathrm{spint}\ (\mathrm{spcl}(S)) \supset S$, a contradiction.

(d) Let $U_1 = \mathrm{spint}\ (\mathrm{spcl}(S))$, where a nonempty $S \in \mathrm{SPO}(X, \tau)$ is such that $U_1 \neq X$. Suppose $U_1 = \emptyset$. Then $\emptyset = \mathrm{spint}(\mathrm{spcl}(S)) \supset S$ [2, Theorem 3.21(a)], a contradiction.

Corollary 5. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is a not* $\mathcal{P}$*-open precontinuous surjection and* (Y, σ) *is submaximal, then* X *allows the partitions* (a) – (d) *from Theorem* 17.

Proof. By Theorem 15.

Definition 3. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* ***pre-α-open*** *if* $f(S) \in \sigma^\alpha$ *for each* $S \in \tau^\alpha$.

Theorem 18. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be open and precontinuous. Then it is pre-α-open.*

Proof. Let a set $A \in \tau^\alpha$ be arbitrarily chosen. By [39, Lemma 4.12(1)] there exists a $U \in \tau$ such that $U \subset A \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(U))$. Since f is open, it follows from [36, Lemma 1.4] that

$f(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))) \subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))))) = \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U)))$. Since f is precontinuous, $\mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(U)))$ [47, Theorem 6.(3)]. Finally, we obtain $f(U) \subset f(A) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(U)))$ and hence, by [39, Lemma 4.12(1)], $f(A) \in \sigma^\alpha$.

The notions of openness and pre-α-openness are independent of each other.

Example 4. (a) *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, *and* $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. *Then, the identity mapping* $\mathrm{id} : (X, \tau) \to (X, \sigma)$ *is open, but it is not pre-α-open since* $\mathrm{id}(\{a, b\}) \notin \sigma^\alpha$.

(b) *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, *and* $\sigma = \{\emptyset, X, \{a\}\}$. *Then, the identity mapping* $\mathrm{id} : (X, \tau) \to (X, \sigma)$ *is pre-α-open and not open since* $\mathrm{id}(\{a, b\}) \notin \sigma$.

A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be ***weakly open*** [46] (resp. ***preopen*** [28]) if $f(U) \subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U)))$ (resp. $f(U) \in \mathrm{PO}(Y, \sigma)$) for every set $U \in \tau$. Each pre-α-open mapping is preopen, but the converse doesn't hold (Example 4(a)).

Theorem 19. *If* $f : (X, \tau) \to (Y, \sigma)$ *is weakly open and precontinuous, then it is preopen.*

Proof. Let $U \in \tau$ be arbitrary. By [47, Theorem 6(3)] we have $f(U) \subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(U)))$.

In [47, Theorem 11], it is shown that preopenness and the so-called ***a.o.W.*** property [50] are equivalent notions. Recall that weak openness and a.o.W. are independent of each other [36, p. 315].

5. α-IRRESOLUTNESS OF MAPPINGS

A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be ***semi-open*** [5] (resp. ***almost open in the sense of Singal and Singal*** [48] or briefly ***a.o.S.***) if $f(U) \in \mathrm{SO}(Y, \sigma)$ (resp. $f(U) \in \sigma$) for every set $U \in \tau$ (resp. $U \in \mathrm{RO}(X, \tau)$). The concepts of preopenness, semi-openness, and a.o.S. are pairwise independent [36].

Lemma 6. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is preopen and precontinuous, then for each set* $S \in \sigma^\alpha$ *there exists a* $U_s \in \sigma$ *with*

(5) $f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$.

Proof. Let an $S \in \sigma^\alpha$ be arbitrarily chosen. By [39, Lemma 4.12(1)] there exists a set $U_s \in \sigma$ such that $U_s \subset S \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))$. Since f is precontinuous, using [47, Theorem 11] we obtain

$$f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{cl}_\sigma(U_s)))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))).$$

This completes the proof.

Lemma 7. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is semi-open and precontinuous, then for each set* $S \in \sigma^\alpha$ *thee exists a* $U_s \in \sigma$ *that satisfies* (5).

Proof. Let an $S \in \sigma^\alpha$. By [39, Lemma 4.12(1)] there exists a set $U_s \in \sigma$ such that $U_s \subset S \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))$. Since f is precontinuous, we have

$$f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))))).$$

It follows by [39, Lemma 4.14] (or by [2, Theorem 1.5(a)]) that $f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{scl}_\sigma(U_s))))$. From [33, Theorem 2] we infer that $f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$. Inclusions $f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$ complete the proof.

Lemma 7 slightly improves a respective part of the proof of [39, Theorem 4.16].

Lemma 8. *If a bijection* $f : (X, \tau) \to (Y, \sigma)$ *is a.o.S. and precontinuous, then for each set* $S \in \sigma^\alpha$ *there exists a* $U_s \in \sigma$ *that satisfies* (5).

Proof. Let an $S \in \sigma^\alpha$. There exists a set $U_s \in \sigma$ such that $U_s \subset S \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))$. We have $f^{-1}(U_s) \subset \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))))$ since by hypothesis $f^{-1}(U_s) \in \mathrm{PO}(X, \tau) \subset \mathrm{SPO}(X, \tau)$. Put $F = Y \setminus f(X \setminus \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))))$. Hence $f(X \setminus f^{-1}(U_s)) \supset f(X \setminus \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))))) = Y \setminus F$. It implies that $F \supset Y \setminus (f(X) \setminus U_s) = U_s$. The set F is closed in (Y, σ), because $\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))) \in \mathrm{RC}(X, \tau)$ and f is a.o.S. Thus, $\mathrm{cl}_\sigma(U_s) \subset F$. Furthermore, we have $f^{-1}(F) = \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))))$. Finally, we obtain $f^{-1}(U_s) \subset f^{-1}(S) \subset f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{cl}_\sigma(U_s)))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(F))) = \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$. This completes the proof.

Every α-irresolute mapping is semi-continuous but even a continuous mapping must not be α-irresolute [27, Example 1]. Every α-irresolute mapping is precontinuous and the converse is not necessarily true [4, Remark].

Theorem 20. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be precontinuous, semi-continuous* (*equiv. α-continuous* [39, Theorem 3.2]), *and let* f *be either*

(a) *preopen* or

(b) *semi-open* [39, Theroem 4.16].

Then f *is α-irresolute.*

Proof. Consider by turns the inclusions (5) from Lemmas 6, 7:

$$f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))),$$

where $S \in \sigma^\alpha$ and $U_s \in \sigma$. Since f is semi-continuous, $\mathrm{cl}_\tau(f^{-1}(U_s)) = \mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(U_s)))$ [34, Lemma 2]. Thus

$$\mathrm{int}_\tau(f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(U_s)))).$$

By [39, Lemma 4.12(1)], f is α-irresolute.

Remark 5. Obviously, Lemmas 6, 7 and Theorem 20 hold if we replace 'precontinuous' by 'α-precontinuous'. In [4, Examples 2.1 & 2.2] it has been shown that α-precontinuity and semi-continuity are independent notions.

Noiri has established that each a.o.S. and α-continuous mapping is α-irresolute [39, Theorem 4.13]. Since the proof of this result is not clear (on a certain stage), we shall prove it in a different way.

Lemma 9. *For a mapping* $f : (X, \tau) \to (Y, \sigma)$ *the following statements are equivalent:*

(a) f *is* α-*irresolute.*

(b) $f\left(cl_{\tau^\alpha}(A)\right) \subset Cl_{\sigma^\alpha}(f(A))$ *for each* $A \subset X$.

(c) $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}_{\sigma^\alpha}(f(A))$ *for each* $A \subset X$.

Proof. (a) $\Leftrightarrow$ (b). Obvious.

(b) $\Leftrightarrow$ (c). Apply [2, Theorem 1.5(c)].

Proof of [39, Theorem 4.13]. We have $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}(f(A))$ for each subset $A \subset X$, because f is α-continuous [30, Theorem 1.1(iv)]. Since f is a.o.S., we get

$$\mathrm{cl}(f(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}(\mathrm{int}(f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))))) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f(A)))).$$

Utilizing [30, Theorem 1.1(iv)] once more, one easily obtains that $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f(A)))) \cup f(A)$ for each $A \subset X$. So, by [2, Theorem 1.5(c)] and Lemma 9(c), f is α-irresolute.

6. HAUSDORFFNESS AND NORMALITY OF SPACES

Definition 4. *A topological space* (X, τ) *is said to be* ***p-Hausdorff*** *if for each pair of distinct points* $x, y \in X$ *there exist disjoint sets* $U_x, U_y \in \mathrm{PO}(X, \tau)$ *with* $x \in U_x$ *and* $y \in U_y$.

Each Hausdorff space is p-Hausdorff. The converse is false in general, as the following example shows.

Example 5. *Consider a space* (X, τ) *where* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$.

The concepts of p-hausdorffiness and the so-called ***semi-Hausdorffness*** [26], are independent of each other. It is evident by Example 5 and [26, Example 4.1].

Theorem 21. *Let $f : (X, \tau) \to (Y, \sigma)$ be a precontinuous injection. If (Y, σ) is Hausdorff then (X, τ) is p-Hausdorff.*

Proof. Omitted.

The results we complete this section with, hold also in some other cases; in particular, for completely continuous mappings (see Remark 2 and [40, Diagram p.249]).

Theorem 22. *If $f : (X, \tau) \to (Y, \sigma)$ is an a.c.S. injection and (Y, σ) is Hausdorff, then (X, τ) is Hausdorff.*

Proof. Use [48, Theorem 2.2(b)] and [10, Lemma 4].

Lemma 10. *A Hausdorff space (X, τ) is normal if and only if for each pair of disjoint sets, $F_1, F_2 \in \mathrm{c}(X, \tau)$ there exist disjoint $U_1, U_2 \in \mathrm{RO}(X, \tau)$ with $F_1 \subset U_1$ and $F_2 \subset U_2$.*

Proof. Similar to that of [10, Lemma 4].

Theorem 23. *Let $f : (X, \tau) \to (Y, \sigma)$ be an a.c.S. closed injection. If (Y, σ) is normal then (X, τ) is normal.*

Proof. Use [48, Theorem 2.2(b)] and Lemma 10.

Recently, the author has introduced the concept of closed-open mappings [10, Definition 1].

Definition 5. *A mapping $f : (X, \tau) \to (Y, \sigma)$ will be called* ***closed-α-open*** *if the image $f(F) \in \sigma^\alpha$ for each $F \in \mathrm{c}(X, \tau)$.*

Each closed-open mapping is closed-α-open. The converse doesn't hold.

Example 6. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$. Define the $f : (X, \tau) \to (X, \tau)$ via $f(a) = c$, $f(b) = a$. $f(c) = b$. Then, f is closed-α-open but not closed-open, because $f(\{b, c\}) \in \tau^\alpha \backslash \tau$.*

Theorem 24. *Let $f: (X, \tau) \to (Y, \sigma)$ be an a.c.S. closed-α-open injection. If (Y, σ) is Hausdorff, then (X, τ) is normal.*

Proof. Hausdorffness of (X, τ) is clear by Theorem 22. Let $F_1, F_2 \in \mathrm{c}(X, \tau)$ be disjoint. We have $f(F_1) \cap f(F_2) = \emptyset$ and there exist sets $G_i \in \sigma$ such that $G_i \subset f(F_i) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_i))$, $i = 1, 2$ [39, Lemma 4.12(1)]. Since $G_1 \cap G_2 = \emptyset$, it follows that

$$\mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_1)) \cap \mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_2)) = \emptyset.$$

The sets $U_i = \mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_i)) \in \mathrm{RO}(Y, \sigma)$, $i = 1, 2$. Therefore, we obtain for both i's that $F_i \subset f^{-1}(U_i) \in \tau$ [48, Theorem 2.2(b)] and $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$. This shows that (X, τ) is normal.

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[1] The reader is requested to use the correct official spelling of author's name, i.e., Duszyński.

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WEAKLY CONTRA-β-CONTINUOUS FUNCTIONS AND STRONGLY Sβ-CLOSED SETS

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ABSTRACT : A new form of contra-β-continuity, called weak contra-β-continuity, is introduced. We show that this class of function is weaker than most forms of β-continuity and contra-continuity, but that the class still has interesting properties. The notion of a strongly Sβ-closed space is defined and conditions are established under which the weakly contra-β-continuous image of a strongly Sβ-closed space is compact. Various properties of strongly Sβ-closed spaces are investigated. For example, we show that every strongly Sβ-closed is the β-closure of a finite set.

Key words and phrases : contra-continuity, β-continuity, contra-β-continuity, weakly contra-β-continuity, nearly compact spaces, strongly Sβ-closed spaces.
2000 Mathematics Subject Classification: 54C10, 54D10.

1. INTRODUCTION

The notion of a β-continuous function was introduced by Abd El-Monsef, et al. [1] in 1983. Over the intervening years many variations of β-continuity have been developed. For example, Noiri and Popa [16] developed the concept of weak β-continuity in 2000 and Noiri [15] introduced slightly β-continuous functions in 2001. Also much work has been done on the notion of contra-continuity and its variations. Contra-continuity was developed by Dontchev [8] in 1996. Dontchev investigated relationships between contra-continuity and coverings of spaces by closed sets. Specifically he defined the notion of a strongly S-closed space and developed conditions under which contra-continuous images of S-closed and strongly S-closed spaces are compact. Caldas and Jafari [6] introduced contra-β-continuity in 2001 and Jafari and Noiri [10] developed contra-precontinuity in 2002. Recently Baker [3, 4, 5] introduced contra-almost β-continuity, weak contra-continuity and weak contra-precontinuity. The purpose of this note is to introduce a form of contra-continuity, which we call weak contra-β-continuity, and to investigate relationships between weak contra-beta-continuity and covers of spaces by β-closed sets. We show weak contra-β-continuity is weaker than β-continuity, contra-β-continuity and weak contra-precontinuity but stronger than slight β-continuity. The concept of a strongly Sβ-closed space is developed and conditions are established under which weakly contra-β-continuous images of strongly Sβ-closed spaces and nearly compact spaces are compact.

2. PRELIMINARIES

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is regular open (respectively, β-open [1], α-open [14], semi-open [11], preopen [13]) provided that $A = \mathrm{Int}(\mathrm{Cl}(A))$ (respectively $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$). A set is regular closed (respectively, β-closed, α-closed, semi-closed, preclosed) if its complement is regular open (respectively, β-open, α-open, semi-open, preopen). The β-closure of a set A, denoted by $\beta\mathrm{Cl}(A)$, is the intersection of all β-closed sets containing A. The preclosure of a set A, denoted by $p\mathrm{Cl}(A)$, is defined analogously using preclosed sets.

Definition 1. *A function $f : X \to Y$ is said to be contra-continuous* [8] *if $f^{-1}(V)$ is closed for every open subset V of Y.*

Definition 2. *A function $f : X \to Y$ is said to be β-continuous* [1] *if $f^{-1}(V)$ is β-open for every open subset V of Y.*

Definition 3. *A function $f : X \to Y$ is said to be contra-β-continuous* [6] *if $f^{-1}(V)$ is β-closed for every open subset V of Y.*

Definition 4. *A function $f : X \to Y$ is said to be weakly contra-continuous* [4] *provided that, whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open Y, then $Cl(f^{-1}(A)) \subseteq f^{-1}(V)$.*

Definition 5. *A function $f : X \to Y$ is said to be weakly contra-precontinuous* [5] *provided that, whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open in Y, then $pCl(f^{-1}(A)) \subseteq f^{-1}(V)$.*

Definition 6. *A function $f : X \to Y$ is said to be slightly β-continuous* [17] *provided that, for every $x \in X$ and every clopen subset V of Y containing $f(x)$, there exists a β-open subset U of X such that $x \in U$ and $f(U) \subseteq V$.*

3. WEAKLY CONTRA-β-CONTINUOUS FUNCTIONS

We define a function $f : X \to Y$ to be weakly contra-β-continuous provided that, if $A \subseteq V \subseteq Y$, with A closed in Y and V open in Y, then $\beta\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$. Since $\beta\mathrm{Cl}(f^{-1}(A)) = f^{-1}(A) \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(A))))$, we have the following characterization of weak contra-β-continuity.

Theorem 3.1. *A function $f : X \to Y$ is weakly contra-β-continuous if and only if, whenever A is a closed subset of Y, V is an open subset of Y, and $A \subseteq V$, then $Int(Cl(Int(f^{-1}(A)))) \subseteq f^{-1}(V)$.*

Theorem 3.2. *If $f : X \to Y$ is β-continuous, then f is weakly contra-β-continuous.*

Proof. Assume $f : X \to Y$ is β-continuous and let $A \subseteq V \subseteq Y$, with A closed in Y and V open in Y. Then, since $f^{-1}(A)$ is β-closed, we have $\beta\mathrm{Cl}(f^{-1}(A)) = f^{-1}(A) \subseteq f^{-1}(V)$, which proves that f is weakly contra-β-continuous.

Theorem 3.3. *If $f : X \to Y$ is contra-β-continuous, then f is weakly contra-β-continuous.*

Proof. Assume $f : X \to Y$ is contra-β-continuous and let $A \subseteq V \subseteq Y$, with A closed in Y and V open in Y. Then, since f is contra-β-continuous, $f^{-1}(V)$ is β-closed. Therefore $\beta\mathrm{Cl}(f^{-1}(A)) \subseteq \beta\mathrm{Cl}(f^{-1}(V)) = f^{-1}(V)$ and hence f is weakly contra-β-continuous.

Since contra-β-continuity and β-continuiy are independent [6], it follows that weak contra-β-continuity does not imply either β-continuity or contra-β-continuity.

The following result is a partial converse of Theorem 3.3.

Definition 7. *A function $f : X \to Y$ is said to be subcontra-β-continuous provided there exists an open base $\mathscr{B}$ for the topology on Y such that $f^{-1}(V)$ is β-closed for every $V \in \mathscr{B}$.*

Recall that a space is called zero dimensional provided it has a clopen base.

Theorem 3.4. *If $f : X \to Y$ is weakly contra-β-continuous and Y is zero dimensional, then f is subcontra-β-continuous.*

Proof. Let $\mathscr{B}$ be a clopen base for Y and let $B \in \mathscr{B}$. Since f is weakly contra-β-continuous, we see that $\beta\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(B)$ and hence that $f^{-1}(B)$ is β-closed.

Since $\beta\mathrm{Cl}(A) \subseteq \mathrm{pCl} \subseteq \mathrm{Cl}(A)$ for every set A, weak contra-continuity ⇒ weak contra-precontinuity ⇒ weak contra-β-continuity. From [5] weak contra-precontinuity does not imply weak contra-continuity and the following example shows that weak contra-β-continuity does not imply weak contra-precontinuity.

Example 3.5. *Assume the space $X = \{a, b, c\}$ has the topologies $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}$ and $\tau = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$. Then the identity mapping $f : (X, \sigma) \to (X, \tau)$ is weakly contra-β-continuous but not weakly contra-precontinuous.*

Theorem 3.6. *If $f : X \to Y$ is weakly contra-β-continuous, then f is slightly β-continuous.*

Proof. Let V be a clopen subset of Y. Then we have $V \subseteq V \subseteq Y$ and, since f is weakly contra-β-continuous, $\beta\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is β-closed, which proves that f is slightly β-continuous.

The following example shows that the implication in Theorem 3.6 can not be reversed.

Example 3.7. [5] *Let X denote the real numbers, let $\sigma = \{X, \emptyset, \{0\}\}$, and let τ be the usual topology on X. Since (X, τ) is connected, the identity mapping $f : (X, \sigma) \to (X, \tau)$ is slightly β-continuous. However, since $\{0\} \subseteq (-1, 1) \subseteq (X, \tau)$ with $\{0\}$ closed and $(-1, 1)$ open, but $\beta Cl(f^{-1}(\{0\})) \not\subseteq f^{-1}(-1, 1)$, f fails to be weakly contra-β-continuous.*

We have the following implications, none of which are reversible:

$$\begin{tikzcd}
 & \text{Weak contra-continuity} \arrow[d, Rightarrow] & \\
 & \text{weak contra-precontinuity} \arrow[d, Rightarrow] & \\
\beta\text{-continuous} \arrow[r, Rightarrow] & \text{weak contra-}\beta\text{-continuous} \arrow[r, Rightarrow] & \text{slight } \beta\text{-continuity} \\
 & \text{contra-}\beta\text{-continuity} \arrow[u, Rightarrow] &
\end{tikzcd}$$

We now investigate relationships between weak contra-β-continuity and various forms of β-continuity and a type of generalized β-continuity.

Definition 8. *A function $f : X \to Y$ is said to be contra-almost β-continuous* [3] *provided that $f^{-1}(V)$ is β-closed for every regular open subset V of Y.*

Recall that a space is extremally disconnected (briefly an ED space) provided that closures of open sets are open.

Theorem 3.8. *If $f : X \to Y$ is weakly contra-β-continuous and Y is an ED space, then f is contra-almost β-continuous.*

Proof. Let V be a regular open subset of Y. Since Y is an ED space, V is also closed. Then, because f is weakly contra-β-continuous, $\beta\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is β-closed, which proves that f is contra-almost β-continuous.

Definition 9. *A function $f : X \to Y$ is said to be weakly β-continuous* [17] *provided that, for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists a β-open subset U of X with $x \in U$ such that $f(U) \subseteq Cl(V)$.*

Since contra-almost β-continuity implies weak β-continuity [3] we have the following corollary.

Corollary 3.9. *If $f : X \to Y$ is weakly contra-β-continuous and Y is an ED space, then f is weakly β-continuous.*

Definition 10. *A subset A of a space X is said to be generalized β-closed (briefly gβ-closed) (also called generalized semi-preclosed)* [7] *provided that $\beta Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*

Definition 11. *A function $f : X \to Y$ is called generalized β-continuous (briefly gβ-continuous) (also called gsp-continuous)* [7] *if $f^{-1}(V)$ is gβ-closed in X for every closed subset V of Y.*

Definition 12. *A function $f : X \to Y$ is said to be approximately β-continuous (briefly aβ-continuous) provided that $\beta Cl(A) \subseteq f^{-1}(V)$ whenever V is open in Y, A is gβ-closed in X, and $A \subseteq f^{-1}(V)$.*

Theorem 3.10. *If $f : X \to Y$ is gβ-continuous and aβ-continuous, then f is weakly contra-β-continuous.*

Proof. Assume $A \subseteq V \subseteq Y$, where A is closed in Y and V is open in Y. Since f is gβ-continuous, $f^{-1}(A)$ is gβ-closed. Thus, because $f^{-1}(A) \subseteq f^{-1}(V)$ and f is aβ-continuous, $\beta\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$, which proves that f is weakly contra-β-continuous.

Theorem 3.11. *If $f : X \to Y$ is weakly contra-β-continuous and images of gβ-closed sets are closed, then f is aβ-continuous.*

Proof. Let V be an open subset of Y, and let A be a gβ-closed subset of X such that $A \subseteq f^{-1}(V)$. Then $f(A)$ is closed, and $f(A) \subseteq V$. Since f is weakly contra-β-continuous, $\beta\mathrm{Cl}(f^{-1}(f(A))) \subseteq f^{-1}(V)$. Then we have $\beta\mathrm{Cl}(A) \subseteq \beta\mathrm{Cl}(f^{-1}(f(A))) \subseteq f^{-1}(V)$, which proves that f is aβ-continuous.

4. STRONGLY Sβ-CLOSED SETS

Thompson [19] defines a spaces to be S-closed provided that every cover of the space by semi-open sets has a finite subfamily, the closures of whose members cover the space. From [9] a space is S-closed if and only if every cover of the space by regular closed sets has finite subcover. Dontchev [8] defines a space to be strongly S-closed provided that every cover of the space by closed sets has a finite subcover. Here we investigate the analogous definitions for β-closed sets.

Definition 13. *A space X is said to be nearly-compact* [18] *if every open cover of X has a finite subfamily such that the interiors of its closures cover X or equivalently if every cover of X by regular open sets has a finite subcover.*

The following characterization of near-compactness will be useful.

Lemma 4.1. *A set U is α-open if and only if $\beta Cl(U) = Int(Cl(Int(U)))$.*

Theorem 4.2. *A space X is nearly-compact if and only if every cover of X by α-open sets has a finite subfamily, the β-closures of whose members cover X.*

Proof. Assume X is nearly-compact. Let C be a cover of X by α-open sets. Then for every $U \in \mathcal{C}$, $U \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. Therefore $\{\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) : U \in \mathcal{C}\}$ is a cover of X by regular open sets and hence has a finite subcover $\{\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U_i))) : i = 1, ..., n\}$. It then follows from Lemma 4.1 that $\{\beta\mathrm{Cl}(U_i) : i = 1, ..., n\}$ is a cover of X.

Assume that every cover of X by α-open sets has a finite subfamily, the β-closures of whose members cover X. Let $\mathcal{C}$ be a cover of X by regular open sets. Then, since $\mathcal{C}$ is a cover of X by α-open sets, there exists a finite subfamily $\{U_i : i = 1, ..., n\}$ such that $\{\beta\mathrm{Cl}(U_i) : i = 1, ..., n\}$ is a cover of X. However, since regular open sets are β-closed, $\beta\mathrm{Cl}(U_i) = U_i$. Hence $\{U_i : i = 1, ..., n\}$ is a finite subcover of $\mathcal{C}$, which proves that X is nearly-compact.

Definition 14. *A space X is said to be strongly $S\beta$-closed if every cover of X by β-closed sets has a finite subcover.*

Theorem 4.3. *If X is strongly $S\beta$-closed, then X is nearly-compact.*

Proof. Let $\mathcal{C}$ be a cover of X by α-open sets. Then for every $U \in \mathcal{C}$, $U \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. Therefore $\{\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) : U \in \mathcal{C}\}$ is a cover of X. It then follows from Lemma 4.1 that $\{\beta\mathrm{Cl}(U) : U \in \mathcal{C}\}$ is a cover for X by β-closed sets. Since X is strongly Sβ-closed, there exists a finite subcover $\{\beta\mathrm{Cl}(U_i) : i = 1,..., n\}$ and hence by Theorem 4.2 X is nearly compact.

Remark 4.4. *Our original intention was to use the condition given in Theorem 4.2 as a definition of $S\beta$-closed. However, this condition turned out to be equivalent to the already defined notion of near-compactness.*

We have the following implications:

strongly Sβ-closed ⇒ strongly S-closed ⇒ S-closed
⇓
nearly-compact

The following examples show that none of the above implications are reversible.

Example 4.5. *Let X denote the real numbers with the topology $\tau = \{U \subseteq X : 0 \notin U$ or $U = X\}$. Since $\{\{x, 0\} : x \neq 0\}$ is a cover of X by regular closed sets with no finite subcover, X fails to be S-closed. However, since X is compact, X is also nearly-compact.*

Example 4.6. *Let X denote the real numbers with the indiscrete topology. Obviously X is strongly S-closed. However, since every subset of X is β-closed, X is not strongly $S\beta$-closed.*

Finally, Dontchev [8] established that strongly S-closed is strictly stronger than S-closed.

Next we investigate the relationship between strong Sβ-closure and compactness. Since Dontchev [8] showed that strong S-closure is independent of compactness, it follows that compactness does not imply strong Sβ-closure. The following example shows strong Sβ-closure is, indeed, independent of compactness.

Example 4.7. *Let X denote the real numbers with the topology $\tau = \{U \subseteq X : 0 \in U$ or $U = \emptyset\}$. Since the β-closed sets coincide with the closed sets, X is strongly $S\beta$-closed. However, X is not compact, since the open cover $\{\{x, 0\} : x \neq 0\}$ has no finite subcover.*

Theorem 4.8. *If X is strongly $S\beta$-closed, there exists a finite set S such that $X = \beta Cl(S)$.*

Proof. The collection $\{\beta\mathrm{Cl}(\{x\}) : x \in X\}$ is a cover of X by β-closed sets. Since X is strongly Sβ-closed, there exists a finite subcover $\{\beta\mathrm{Cl}(\{x_i\}) : i = 1, ..., n\}$. Then $X = \cup_{i=1}^{n}\beta\mathrm{Cl}\{x_i\} \subseteq \beta\mathrm{Cl}(\{x_1, x_2,...,x_n\})$.

If we add the hypothesis that no singleton set of X be open to Theorem 4.8, then the singleton sets are β-closed and we obtain the following result.

Theorem 4.9. *A strongly Sβ-closed space in which the singleton sets are not open is finite.*

The proof of the following theorem is straightforward.

Theorem 4.10. *If $f : X \to Y$ is contra-β-continuous and X is strongly Sβ-closed, then $f(X)$ is compact.*

Definition 15. *A space X called a C-space provided that, for every open set U of X and every $x \in U$, there exists a closed set A such that $x \in A \subseteq U$.*

Theorem 4.11. *Let $f : X \to Y$ be weakly contra-β-continuous, and let Y be a C-space. If X is strongly Sβ-closed, then $f(X)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$ by open subsets of X. Let $y \in f(X)$ and $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is a C-space, there exists a closed set A_y such that $y \in A_y \subseteq V_y$. Because f is weakly contra-β-continuous, $\beta\mathrm{Cl}(f^{-1}(A_y)) \subseteq f^{-1}(V_y)$. Then we see that $\{\beta\mathrm{Cl}(f^{-1}(A_y)) : y \in f(X)\}$ is a cover of X by β-closed sets. Since X is strongly Sβ-closed, there exists a finite subcover $\left\{\beta\mathrm{Cl}\left(f^{-1}\left(A_{y_i}\right)\right) : i = 1, 2, \ldots, n\right\}$. It follows that $\left\{V_{y_i} : i = 1, 2, \ldots, n\right\}$ is a finite subcover of $\mathcal{C}$ and hence that $f(X)$ is compact.

Definition 16. *A function $f : X \to Y$ is said to be contra-α-continuous if, for every open set V of Y, $f^{-1}(V)$ is α-closed.*

Theorem 4.12. *Let $f : X \to Y$ be weakly contra-β-continuous and contra-α-continuous and assume that Y is a C-space. If X is nearly-compact, then $f(X)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$ by open subsets of X and let $y \in f(X)$. Then let $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is a C-space, there exists a closed set A_y such that $y \in A_y \subseteq V_y$. Since f is contra-α-continuous, $f^{-1}(A_y)$ is α-open. It follows that the family $\{f^{-1}(A_y) : y \in f(X)\}$ is a cover of X by α-open sets. Since Y is nearly-compact, by Theorem 4.2 there exists a finite subfamily $\left\{f^{-1}\left(A_{y_i}\right) : i = 1, \ldots, n\right\}$ such that $X = \cup_{i=1}^{n} \beta\mathrm{Cl}\left(f^{-1}\left(A_{y_i}\right)\right)$.

Because f is weakly contra-β-continuous, $\beta\mathrm{Cl}\left(f^{-1}\left(A_{y_i}\right)\right) \subseteq f^{-1}\left(V_{y_i}\right)$ for every i. Thus $\left\{V_{y_i} : i = 1, \ldots, n\right\}$ is a finite subcover of $\mathcal{C}$ and we see that $f(X)$ is compact.

Remark 4.13. *The combination of weak contra-β-continuity and a C-space for a codomain does not imply contra-β-continuity. Consider the identity mapping on the real numbers with the usual topology.*

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CROSSING AND THICKNESS OF SPECIAL TYPE OF NON-PLANAR GRAPH

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ABSTRACT : In this paper, we have constructed some non-planar graphs from the TEEP graph and studied various properties of them, relating to the thickness and crossing. In addition to this, the edge-disjoint Hamiltonian circuit of the non-planar graphs has been discussed.

AMS Subject-Classification: 05C30, 05C45, 05c62, 05c83 (MSC 2000)

1. INTRODUCTION

The thickness and crossings of non-planar graph play very important role in case of printed circuit board. It is important to minimize the wire crossings in case of VLSI design technology, particularly in circuit layout process. There are graphs, namely Hyper graph, clique graph, interval graph, circuit graph etc. which are directly or indirectly related with VLSI design in floor plan. [12]. Many problems in this area are still remaining unsolved. It is known that the thickness and crossing of a planar graph is one and zero respectively. The thickness and crossings of a few non-planar graphs have been found [1-7]. Recently Kalita B, [8] found a set of non-planar graph, and studied the maximum and minimum number of crossings. He further cited some examples with corssings of non-planar graph. The upper bound of crossing of complete graph Km and bipartite graph Km, n is found by the theorem (11.23) [13]. The problem of finding the crossings and thickness of an arbitrary non-planar graph is not found till today. There are some particular types of graphs in which crossings and thickness can be found.

In this paper, we have developed some non-planar graphs and studied the crossings and thickness of them. The non-planar graphs have been constructed from TEEP graph [8,14]. The paper is organized as follows. In section 1, we have explained some works of crossing and thickness of graph. In section 2, the notation and terminology are considered. The definition and construction of non-planar graphs are included in section 3. Section 4 contains the theoretical explanation and properties. The conclusion is included in section 5.

2. NOTATIONS AND TERMINOLOGY

The notation and terminology are considered from the standard reference [1-14]. The thickness and crossings of non-planar graph G are denoted by θ (G) and Φ (G). The maximum and minimum degrees are denoted by Δ and ∂ respectively.

3. DEFINITION AND CONSTRUCTION OF NON-PLANAR GRAPH

Before going to construct the non-planer graph from TEEP graphs, we remind the definition of crossings and thickness of non-planar graph.

3.1. Crossing. The minimum number of crossing in a drawing of a non-planar graph G in a plane is called the crossing number. It is denoted by Φ (G).

3.2. Thickness. The thickness of a non-planar graph G is the minimum number of planar graphs in a decomposition of G into planar sub-graphs graphs. It is denoted by θ (G).

3.3. Construction of non-planar graph from TEEP graph. Let G (2m + 2,6m) and H (2m+3, 6m+3) for m $\geq$ 2 are two TEEP graphs [14]. These two graphs have minimum degree $\delta = 3$ and maximum degree $\Delta = 2m + 1$ and $\Delta = 2m + 2$ for m $\geq$ 2 respectively. There are two vertices of degree four each, two vertices of degree three and the remaining vertices are of degree five. The following two figures [fig 1 & fig 2] give the TEEP graph G & H for m = 2.

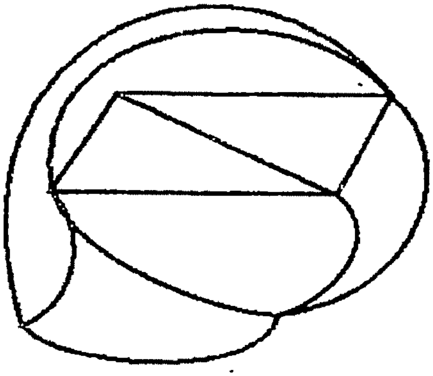

fig-1

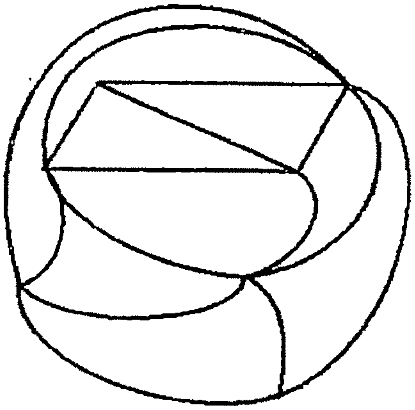

fig-2

Now joining the two vertices of minimum degree by one edge of the graph G (2m + 2, 6m) for m $\geq$ 2 [similarly for the graph H (2m + 3,6m + 3) for m $\geq$ 2], and continuing the process of joining the other vertices of minimum degree of the graph G (2m + 2,6m)/ H (2m + 3, 6m + 3) for m $\geq$ 2 by one edge till the graph thus obtained forms a non-planar graph which is the preceding to the complete graph K_{2m-2}/K_{2m+3} for $m \geq 2$. Thus, we have two non-planar graph $G'(2m + 2, 6m + 2)$ and H' $(2m + 3, 6m + 8)$ for $m \geq 2$. The figure (3) and (4) shows the non-planar graphs G' (6.14) and H' (7, 20) for $m = 2$ which are the preceding one of the complete graph K_{2m+2}/K_{2m+3}.

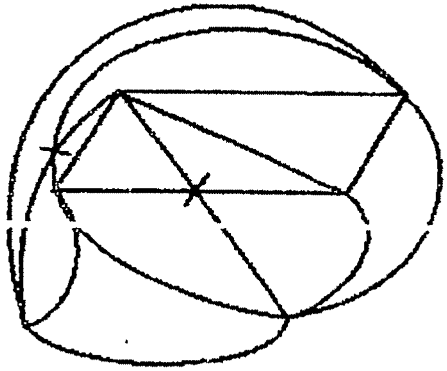

fig-3

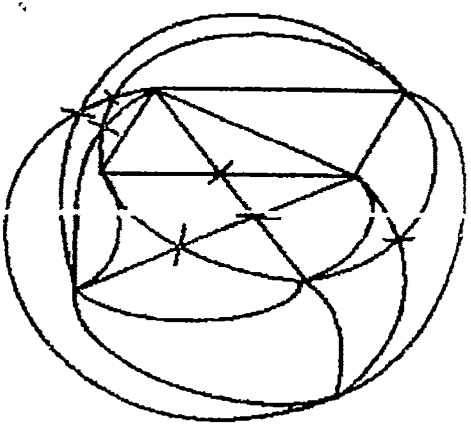

fig-4

These two graphs do not satisfy the conditions of planarity that is the number of edges is not les than or equal to $3n - 6$, where n is the number of vertices of the graph. Hence we have the following two inequalities from the construction process of the non-planar graphs drawn earlier as

$$6m + 2 \nleq 3(2m + 2) - 6 \text{ and}$$

$$6m + 8 \nleq 3(2m + 3) - 6 \text{ for } m \geq 2.$$

Remarks: The non-planar graphs constructed in this paper as shown earlier in figure –3, figure –4, which is completely different types of non-planar graph drawn by Kalita [8].

4. THEORETICAL EXPLANATION AND PROPERTIES

4.1. Theorem. For the non-planar graph $G'(2m + 2, 6m + 1)$ for $m = 2, 3, 4\ldots$ the number of crossings is $m - 1$ that is $\Phi(G) = m - 1$.

Proof. We know that the TEEP graph $(2m + 2, 6m)$ for $m \geq 2$ [14] is a planar graph. WE have constructed the graph $G'(2m + 2, 6m + 1)$ for $m \geq 2$ by joining the vertices of minimum degree $\delta = 3$ of the graph $G(2m + 2, 6m)$ by one edge. Definitely the $G'(2m + 2, 6m + 1)$ for $m \geq 2$ is a non-planar as we have

$$6m + 1 \nleq 3(2m + 2) - 6$$

$$\Rightarrow 6m + 1 \nleq 6m \text{ for } m \geq 2.$$

Now we have to prove that the number of crossings of the graph G′(2m + 2, 6m + 1) is (m − 1), Φ(G′) = m − 1.

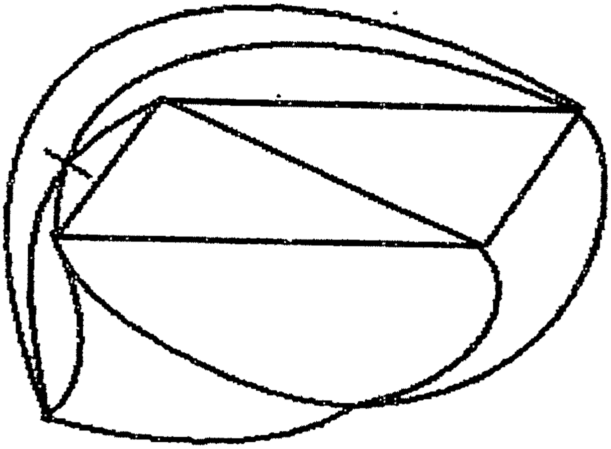

fig-5

Let us prove it by the method of induction. It is observed that for m = 2 the crossings number is one. [From fig-5] Φ(G′) = 1. Which is true for m = 2.

Now for m = 3 it can be shown that the number of crossings is two.

Let us consider that the above theorem is true for m = K, where the number of crossings is K − 1. I.e. the crossings of G′(2K + 2, 6K + 1) is (K − 1). For the next stage we proceed by adding two vertices and six edges we have the structure G′(2K + 2 + 2, 6K + 1 + 6) and the number of crossings are (K − 1 + 1) i.e. G′[2(K + 1) + 2, 6(k + 1) + 1] the crossing number is K. Hence the theorem is true for m = K + 1. i.e. the number of crossings for m = K + 1 is K. Which is true for m = K + 1. Hence the theorem is true for all m ≥ 2. Now we can conclude that the number of crossings of non-planar graph G′(2m + 2, 6m + 1) for m ≥ 2 is (m − 1).

I.e. Φ (G) = m − 1.

4.2. Theorem. The crossings of non-planar graph H′(2m + 3, 6m + 4) for m ≥ 2 is m i.e. Φ(H′) = m.

Proof. We have for the TEEP graph H(2m + 3, 6m + 3) for m ≥ 2, the minimum degree δ = 3 and maximum degree Δ = 2m + 2. There are two vertices of minimum degree ∂ = 3 and two vertices of degree four and the remaining vertices are of degree five. Now we have to prove that the number of crossings of the graph H′(2m + 3, 6m + 4) for m ≥ 2 is m.

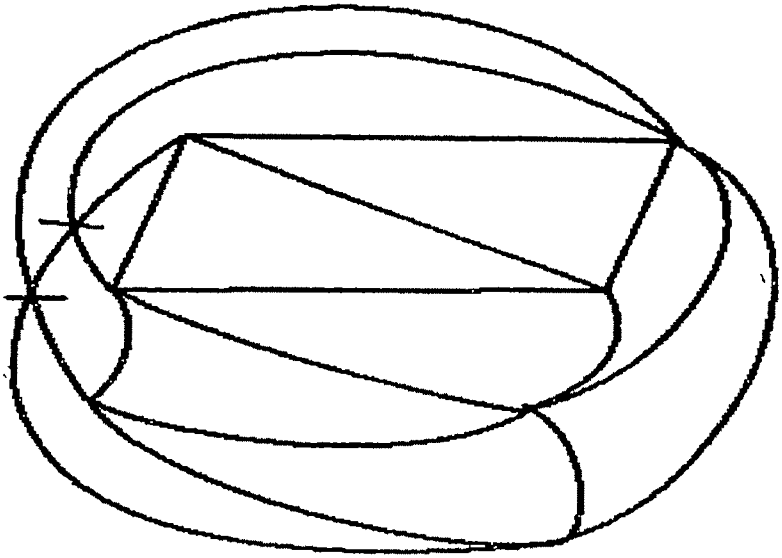

fig. 6

Let us prove it by the method of induction. It is observed that for m = 2 the graph H′(7, 16) has two crossings [fig- 6]. Again it can be shown that for m = 3 the graph H′(9, 22) has three crossing. Thus we found that the result is true for m = 2 and m = 3.

Let us considêr that the above theorem is true for m = k. Then the structure of the non-planer graph is H′(2k + 3, 6k + 4) and then the number of crossing is K. Now when we put m = k + 1 then the non-planar graph H′(2K + 3, 6K + 4) takes the from as H′(2K + 5, 6K + 4) for K ≥ 3, Which automatically shows that the crossings of the graph is k + 1. Hence the above theorem is true for all m ≥ 2 i.e. the of crossings of the non-planar graph H′(2m + 3, 6m + 4) for m ≥ 2 are m i.e. Φ(H′) = m.

4.3. Theorem. The difference between the number of crossings of the complete graph K_{2m+2} for m ≥ 2 and the number of crossings of its preceding one (not complete) constructed from the TEEP graph G(2m + 2, 6m) for m ≥ 2 is ≥ (m − 1).

Proof. We have the TEEP graph G(2m + 2, 6m) for m ≥ 2 is a planar graph where the two vertices has minimum degree δ = 3 other two vertices has degree four, one vertex has maximum degree Δ = 2m + 1 but remainging vertices are of degree five as shown earlier in fig. (1) & (2). Now we have to prove the difference between the crossings of complete graph K_{2m+2} and its preceding one is always ≥ (m − 1).

Let us prove it by the method of induction. Now for m = 2, the structure of G(2m + 2, 6m) is of the form G(6, 12) which is clearly a planar graph. But we have

G(6, 15) is a complete graph for m = 2 and G(6, 14) is its preceding one of the complete graph of G(6, 15) as shown in fig 3 earlier. But we know that the number of crossings of the complete graph K_{2m+2} are 3 for m = 2. Now if we delete any one edge from the complete graph, which contain the minimum number of crossings, and then we observe that the number of crossings of the preceding of the complete graph is found to be 2. Thus the difference is (3 – 2) = 1, that is 2 – 1 = 1 which is true for m = 2.

Similarly we suppose that the theorem is true for m = 3. Then we have the difference between the crossings of the complete graph and its preceding one is m – 1 = 2 for m = 3.

Suppose the theorem is true for m = k. Then we find out that the difference between the crossing of the complete graph K_{2k+2} it's preceding one is K – 1. Again if we put the value of m is equal to k + 1 then obviously it is found that the difference between the number of crossings of complete graph and its preceding one as (K + 1) – 1 = k for k ≥ 3 which shows that the result is also true for all values of m ≥ 2. Hence by the method of induction we have the difference between the number of crossings of the complete graph and its preceding one is always ≥ (m – 1).

4.4. Theorem. The difference between the number of crossings of complete graph K_{2m+3} and its preceding one (constructed from the TEEP graph H(2m + 3, 6m + 3) of the complete graphs for m ≥ 2 is at least (2m – 3).

Proof. We know that the TEEP graph H(2m + 3, 6m + 3) for m ≥ 2 is a planar graph. We have construected earlier a non-planar graph as shown in the [fig-4], which is the preceding one of the complete graph K_{2m+3}.

Now we are to prove that the difference between the number of crossings of the complete graph K_{2m+3} and its preceding one is (2m – 3).

We will prove it by the method of induction.

Now for m = 2 the structure H(7, 15), which is nothing but a TEEP graph, and it is a planar graph and H(7, 21) is the complete graph of the above planar graph. Now joining the vertices by one edge in between the two minimum degree vertices continuously till the preceding one of the complete graph is obtained, then we observed that the number of crossings of the preceding one is 8. But, the crossings of the complete graph K_{2m+3} are 9.

Thus we observed that thedifference between the number of crossings of complete graph and its preceding one is (9 – 8) = 1 i.e. (2.2 – 3) = 1 which shows that the theorem is true for m = 2.

Again if we put the value of m equal to 3 then the number of crossings of the complete graph and the number of crossing of preceding complete graph is (2.3 – 3) = 3.

Now we consider that the above theorem is true for $m = K$.

Hence the difference between the number of crossing of complete graph and its preceding one is $2K - 3$.

Now, putting the value of m as $k + 1$ we can show that the relation $2m - 3$ gives as $2(k + 1) - 3 = 2k + 2 - 3 = 2k - 1$ for $k \geq 3$ which is true for $m = k + 1$.

Hence the above theorem is true for all values of $m \geq 2$.

4.5. Theorem. The thickness of the non-planar graph $G'(2m + 2, 6m + 2)$ and $H'(2m + 3, 6m + 8)$ for $m \geq 2$ constructed from the TEEP graphs $G(2m + 2, 6m)$ and $H(2m + 3, 6m + 3)$ is always 2 that is $\theta(G') = \theta(H') = 2$.

Proof. We know that the TEEP graph $G(2m + 2, 6m)$ and $H(2m + 3, 6m + 3)$ for $m \geq 2$ are planar graph and these two graphs have two vertices of minimum degree $\delta = 3$ and one vertex has maximum degree $A = 2m + 1$ and $A = 2m + 2$ respectively. Now from the graph $G(2m + 2, 6m)$ and $H(2m + 3, 6m + 3)$ for $m \geq 2$ by joining vertices of minimum degrees by one edge only and continuiting the process till the preceding of complete graphs $G'(2m + 2, 6m + 2)$ and $H'(2m + 3, 6m + 8)$ are obtained and they are non-planar. But we require finding out the decomposition of these two non-planar graphs as planar sub-graphs and it can be shown that there will be only two such decompositions for theses two non-planar graphs. That is the thickness of the non-planar graphs $G'(2m + 2, 6m + 2)$ and $H'(2m + 3, 6m + 8)$ is always two for $m = 2$.

Now for $m = 3$ we observe that the thickness of the graphs $G'(2m + 2, 6m + 2)$ and $H'(2m + 2, 6m + 5)$ is also 2.

Thus we can conclude that the above theorem is true for $m = 2$ and $m = 3$.

We know that thickness of the complete graphs K_{2m+2} and K_{2m+3} for $m \geq 2$ can be found from the theorems [11.23 and 11.24] [13] and we have that the non-planar graphs $G'(2m + 2, 6m + 2)$ and $H'(2m + 3, 6m + 8)$ for $m \geq 2$ are constructed from the TEEP graphs $G(2m + 2, 6m)$ and $H(2m + 3, 6m + 3)$ for $m \geq 2$ by joining some vertices by some edges as discussed in section 3.3. It can be shown easily that there requires only two plans to minimize the number of crosing of these two graphs for $m \geq 2$. Hence we must comment that the non-planar graph $G'(2m + 2, 6m + 2)$ and $H'(2m + 3, 6m + 8)$ have require only two planes. That is the thickness is two.

4.6. Thorem. The number of edge disjoint Hamiltonian circuits of the non-planar graph $G'(2m + 2, 6m + 2)$ made from the planar graph $G(2m + 2, 6m)$ for $m \geq 2$ up to the preceding of K_{2m+2} is always 2.

Proof. We know that from the TEEP graph G(2m + 2, 6m) for m > 2 which has only one edge joint Hamiltonian Circuit. Now if we joined by one edge in between the minimum degree vertices by one edge and continue the process up to the preceding of complete graph of the mentioned graph then also the edge disjoint Hamiltonian Circuit is also two [from fig (3)]. Because for m = 2 the degree of the vertices up to the preceding one of the complete graph is observed that which has four vertices of maximum degree $\Delta = 5$ and two vertices of minimum degree $\delta = 4$. So it has only two edges disjoint Hamiltonian Circuits.

Which is true for m = 2

Thus trivially we can proved that for all values of $m \geq 2$ the non-planar graph G′(2m + 2, 6m + 1) for $m \geq 2$ which has two edge disjoint Hamiltonian circuits. And continuing the process of joining the vertices by edges up to proceding of the complete graph K_{2m+2} for $m \geq 2$, which has also two edges, disjoint Hamiltonian Circuits.

4.7. Theorem. The number of edge disjoint Hamiltonian Circuits of the non-planar graph made from the planar graph H (2m + 3, 6m + 3) for $m \geq 2$ up to the preceding of K_{2m+3} is always 2.

Proof. The proof of this theorem as [4.6].

5. CONCLUSION

This paper gives a special type of non-planar graphs, which are constructed from the TEEP graphs and the thickness and crossing of these non-planar graphs are found from various theorems. Hence one can study the thickness and crossing of these non-planar graphs for various design purposes, which is lies in the VLSI design technology.

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TRANSFORMATIONS OF SOME MULTIPLE HYPER-GEOMETRIC FUNCTIONS OF SEVERAL VARIABLES*

R. C. SINGH CHANDEL

ABSTRACT : In the present paper, the author applies the difference operators, to establish many transformations of multiple hypergeometric functions of several variables introduced by Chandel et al. ([4] and [5]). He also discusses their special cases.

Key words and pharases : Multiple Hypergeometric Functions of Several Variables, Difference Operators, Transformations.
2000 Mathematics Subject Classification : Primary 47B39; Secondary 33C65.

1. INTRODUCTION

Making an appeal to the difference operators Δ and E, Agrawal [1, 2] and Gupta and Agrawal [7, 8, 9] obtaned various transformations of hypergeometric series. Further, Gupta and Agarwal [10] obtained a transformation of Lauricella's multiple hypergeometric function $F_A^{(n)}$ ([11]; see also [13]).

In the present paper, making an appeal to these difference operators, we obtain various transformations of some multiple hypergeometirc functions of several variables introduced and studied by Chandel et al. ([3] to [5]). We also discuss their special cases, which may be useful in related topics of mathematical analysis.

2. TRANSFORMATION FORMULAS

Consider

$$A = \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)}[1 - x_1E_1 - \ldots - x_nE_n]^{-a}$$

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$$\left\{\frac{\Gamma(b_1)\ldots\Gamma(b_n)}{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}\right\}$$

$$= \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} (a)_{m_1+\ldots+m_n} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$\left\{\frac{\Gamma(b_1+m_1)\ldots\Gamma(b_n+m_n)}{\Gamma(c_1+\ldots+c_k+m_1+\ldots+m_k)\Gamma(c_{k+1}+m_{k+1})\ldots\Gamma(c_n+m_n)}\right\}$$

$$= {}_{(k)}F_{AD}^{(n)}(a, b_1, \ldots, b_n, b_n, c_1+\ldots+c_k, c_{k+1}, \ldots, c_n, x_1, \ldots, x_n)$$

Also we can write

$$A = \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)}\left[1-x_{k+1}-\ldots-x_n\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\ldots+x_kE_k+x_{k+1}\Delta_{k+1}+\ldots+x_n\Delta_n}{1-x_{K+1}-\ldots-x_n}\right]^{-a}\left\{\frac{\Gamma(b_1)\ldots\Gamma(b_n)}{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}\right\}$$

$$= (1-x_{k+1}-\ldots-x_n)^{-a}$$

$${}^{(k)}F_{AD}^{(n)}\Big(a, b_1, \ldots, b_k, c_{k+1}-b_{k+1}, \ldots, c_n-b_n; c_1+\ldots+c_k, c_{k+1}, \ldots c_n$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n}, \ldots, \frac{x_k}{1-x_{k+1}-\ldots-x_n}, \frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n}, \ldots, \frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Big).$$

Equating both values of A, we derive

$$(2.1)\quad {}^{(k)}F_{AD}^{(n)}\left(a,b_1,\dots,b_n;\,c,c_{k+1},\dots,c_n;\,x_1,\dots,x_n\right)$$

$$= \left(1-x_{k+1}-\dots-x_n\right)^{-a}$$

$$\qquad {}^{(k)}F_{AD}^{(n)}\left(a,b_1,\dots,b_k,c_{k+1}-b_{k+1},\dots,c_n-b_n;\,c,c_{k+1},\dots,c_n;\,x_1,\dots,x_n\right)$$

$$\left.\frac{x_1}{1-x_{k+1}-\dots-x_n},\dots,\frac{x_k}{1-x_{k+1}-\dots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\dots-x_n},\dots,\frac{-x_n}{1-x_{k+1}-\dots-x_n}\right)$$

where ${}^{(k)}F_{AD}^{(n)}$ is multiple hypergeometric function related to Lauricella's functions introduced by Chandel and Gupta [4].

Similarly, by considering

$$\frac{\Gamma(c_1)\dots\Gamma(c_n)}{\Gamma(b_1+\dots+b_k)\Gamma(b_{k+1})\dots\Gamma(b_n)}\left[1-x_1E_1-\dots-x_nE_n\right]^{-a}$$

$$\left\{\frac{\Gamma(b_1+\dots+b_k)\Gamma(b_{k+1})\dots\Gamma(b_n)}{\Gamma(c_1)\dots\Gamma(c_n)}\right\},$$

we derive

$$(2.2)\quad {}^{(k)}F_{AC}^{(n)}\left(a,b,b_{k+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n\right)$$

$$= \left(1-x_{k+1}-\dots-x_n\right)^{-a}\,{}^{(k)}F_{AC}^{(n)}\Bigg(a,b,c_{k+1}-b_{k+1},\dots,c_n-b_n;c_1,\dots,c_n;$$

$$\left.\frac{x_1}{1-x_{k+1}-\dots-x_n},\dots,\frac{x_k}{1-x_{k+1}-\dots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\dots-x_n},\dots,\frac{-x_n}{1-x_{k+1}-\dots-x_n}\right).$$

For $k = 0$, (2.1) and (2.2) both give following transformation formula for Lauricella's multiple hypergeometric function $F_A^{(n)}$ [11].

$$(2.3)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$= \left(1-x_1-\ldots-x_n\right)^{-a} F_A^{(n)}\Bigg(a,c_1-b_1,\ldots,c_n-b_n;c_1,\ldots,c_n;$$

$$\frac{-x_1}{1-x_1-\ldots-x_n},\ldots,\frac{-x_n}{1-x_1-\ldots-x_n}\Bigg),$$

which can be verified by choosing $k = 2$ separately in both the results due to Agrawal [1, p. 116] and Gupta and Agrawal [10].

Further, for $k = 1$, both give

$$(2.4)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$= \left(1-x_2-\ldots-x_n\right)^{-a} F_A^{(n)}\Bigg(a,c_1-b_1,\ldots,c_n-b_n;c_1,\ldots,c_n;$$

$$\frac{x_1}{1-x_2-\ldots-x_n},\frac{-x_2}{1-x_2-\ldots-x_n},\ldots,\frac{-x_n}{1-x_2-\ldots-x_n}\Bigg),$$

which can be further extended in the from of n unified results

$$(2.5)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right) = \left(1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n\right)^{-a}$$

$$F_A^{(n)}\Bigg(a,c_1-b_1,\ldots,c_{i-1}-b_{i-1},b_i,c_{i+1}-b_{i+1},\ldots,c_n-b_n;c_1,\ldots,c_n;$$

$$\frac{-x_1}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n},\ldots,\frac{-x_{i-1}}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n},$$

$$\frac{x_i}{1-x_i-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n},\frac{-x_{i+1}}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n}$$

$$\ldots,\frac{-x_n}{1-x_1-\ldots-x_{i-1}-x_{i+1}-\ldots-x_n}\Bigg),\quad i = 1,\ 2,\ \ldots,\ n.$$

Further considering

$$\frac{\Gamma(c_1)\ldots\Gamma(c_n)}{\Gamma(b_{k+1})\ldots\Gamma(b_n)}\left[1-x_1E_1-\ldots-x_nE_n\right]^{-a}\left\{\frac{\Gamma(b_{k+1})\ldots\Gamma(b_n)}{\Gamma(c_1)\ldots\Gamma(c_n)}\right\}$$

we derive

$$(2.6)\ {}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$$

$$= \left(1-x_{k+1}-\ldots-x_n\right)^{-a}\,{}^{(k)}_{(2)}\phi^{(n)}_{AC}\Bigg(a,c_{k+1}-b_{k+1},\ldots,c_n-b_n;c_1,\ldots,c_n;$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{x_k}{1-x_{k+1}-\ldots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Bigg),$$

where ${}^{(k)}_{(2)}\phi^{(n)}_{AC}$ is the confluent form of Karlsson's multiple hypergeometric function introduced by Chandel and Vishwakarma [5].

For $k = 0$. (2.6) further gives (2.3).

3. GENERALIZATION OF (2.1)

Equation (2.1) can also be written as follows:

$$A = \frac{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}{\Gamma(b_1)\ldots\Gamma(b_n)}\left[1-x_{k+1}-\ldots-x_n\right]^{-a}$$

$$\left[1-\frac{x_1E_1+\ldots+x_mE_m+x_{m+1}\Delta_{m+1}+\ldots+x_n\Delta_n}{1-x_{m+1}-\ldots-x_n}\right]^{-a}\left\{\frac{\Gamma(b_1)\ldots\Gamma(b_n)}{\Gamma(c_1+\ldots+c_k)\Gamma(c_{k+1})\ldots\Gamma(c_n)}\right\}$$

$$= \left(1-x_{m+1}-\ldots-x_n\right)^{-a}$$

$${}^{(k)}F^{(n)}_{AD}\Bigg(a,b_1,\ldots,b_m,c_{m+1}-b_{m+1},\ldots,c_n-b_n;c_1+\ldots+c_k,c_{k+1},\ldots,c_n;$$

$$\frac{x_1}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{x_m}{1-x_{m+1}-\ldots-x_n},\frac{-x_{m+1}}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{m+1}-\ldots-x_n}\Bigg),$$

which gives

$$(3.1)\quad {}^{(k)}F_{AD}^{(n)}\left(a,b_1,\ldots,b_n;\,c,c_{k+1},\ldots,c_n;\,x_1,\ldots,x_n\right)$$

$$= \left(1-x_{m+1}-\ldots-x_n\right)^{-a}\,{}^{(k)}F_{AD}^{(n)}\Bigg(a,b_1,\ldots,b_m,c_{m+1}-b_{m+1},\ldots,c_n-b_n;c,c_{k+1},\ldots,c_n;$$

$$\frac{x_1}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{x_m}{1-x_{m+1}-\ldots-x_n},\frac{-x_{m+1}}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{m+1}-\ldots-x_n}\Bigg)$$

where m and k are non-negative integers $\leq n$.

For $m = k$, (3.1) reduces to (2.1).

Replacing k by 0 and m by k, (3.1) reduces to the following result :

$$(3.2)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;\,c_1,\ldots,c_n;\,x_1,\ldots,x_n\right)$$

$$= \left(1-x_{k+1}-\ldots-x_n\right)^{-a} F_A^{(n)}\Bigg(a,b_1,\ldots,b_k,c_{k+1}-b_{k+1},\ldots,c_n-b_n;c_1,\ldots c_n;$$

$$\frac{x_1}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{x_k}{1-x_{k+1}-\ldots-x_n},\frac{-x_{k+1}}{1-x_{k+1}-\ldots-x_n},\ldots,\frac{-x_n}{1-x_{k+1}-\ldots-x_n}\Bigg),$$

which also suggets the following results due to Gupta and Agrawal [10] and Agrawal [2]:

$$(3.3)\quad F_A^{(n)}\left(a,b_1,\ldots,b_n;c_1,\ldots,c_n;\,x_1,\ldots,x_n\right)$$

$$= \left(1-x_1-\ldots-x_n\right)^{-a} F_A^{(n)}\Bigg(a,c_1-b_1,\ldots,c_k-b_k,b_{k+1},\ldots,b_n;\,c_1,\ldots c_n;$$

$$\frac{-x_1}{1-x_1-\ldots-x_k},\ldots,\frac{-x_k}{1-x_1-\ldots-x_k},\frac{x_{k+1}}{1-x_1-\ldots-x_k},\ldots,\frac{x_n}{1-x_1-\ldots-x_k}\Bigg).$$

As for (2.2), again considering

$$ {}^{(k)}F_{AC}^{(n)}\left(a, b_1 + \ldots + b_k, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right) $$

$$ = \frac{\Gamma\left(c_1\right) \ldots \Gamma\left(c_n\right)}{\Gamma\left(b_1 + \ldots + b_k\right)\Gamma\left(b_{k+1}\right) \ldots \Gamma\left(b_n\right)}\left[1 - x_1 E_1 - \ldots - x_n E_n\right]^{-a} $$

$$ \left\{\frac{\Gamma\left(b_1 + \ldots + b_k\right)\Gamma\left(b_{k+1}\right) \ldots \Gamma\left(b_n\right)}{\Gamma\left(c_1\right) \ldots \Gamma\left(c_n\right)}\right\} $$

$$ = \frac{\Gamma\left(c_1\right) \ldots \Gamma\left(c_n\right)}{\Gamma\left(b_1 + \ldots + b_k\right)\Gamma\left(b_{k+1}\right) \ldots \Gamma\left(b_n\right)}\left[1 - x_{M+1} - \ldots - x_n\right]^{-a} $$

$$ \left[1 - \frac{x_1 E_1 + \ldots + x_M E_M + x_{M+1}\Delta_{M+1} + \ldots + x_n \Delta_n}{1 - x_{M+1} - \ldots - x_n}\right]^{-a} $$

$$ \left\{\frac{\Gamma\left(b_1 + \ldots + b_k\right)\Gamma\left(b_{k+1}\right) \ldots \Gamma\left(b_n\right)}{\Gamma\left(c_1\right) \ldots \Gamma\left(c_n\right)}\right\} $$

we establish

$$ (3.4) \quad {}^{(k)}F_{AC}^{(n)}\left(a, b, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right) $$

$$ = \left(1 - x_{M+1} - \ldots - x_n\right)^{-a} $$

$$ {}^{(k)}F_{AC}^{(n)}\left(a, b, b_{k+1}, \ldots, b_M; c_{M+1} - b_{M+1}, \ldots, c_n - b_n; c_1, \ldots, c_n;\right. $$

$$ \left.\frac{x_1}{1 - x_{M+1} - \ldots - x_n}, \ldots, \frac{x_M}{1 - x_{M+1} - \ldots - x_n}, \frac{-x_{M+1}}{1 - x_{M+1} - \ldots - x_n}, \ldots, \frac{-x_n}{1 - x_{M+1} - \ldots - x_n}\right), $$

$(0 \le k \le M \le n$ and $k, M\ n \in l)$.

which, for $M = k$, reduces to (2.2); while for $M = k = 0$, (3.4) reduces to (2.3).

Further, for $k = 0$, (3.4) reduces to

$$(3.5)\quad F_A^{(n)}\left(a, b_1, \ldots, b_n;\, c_1, \ldots, c_r;\, x_1, \ldots, x_n\right)$$

$$= \left(1 - x_{M+1} - \ldots - x_n\right)^{-a} F_A^{(n)}\Bigg(a, b_1, \ldots, b_M;\, c_{M+1} - b_{M+1}, \ldots, c_n - b_n;\, c_1, \ldots, c_n;$$

$$\frac{x_1}{1 - x_{M+1} - \ldots - x_n}, \ldots, \frac{x_M}{1 - x_{M+1} - \ldots - x_n}, \frac{-x_{M+1}}{1 - x_{M+1} - \ldots - x_n}, \ldots,$$

$$\frac{-x_n}{1 - x_{M+1} - \ldots - x_n}\Bigg).$$

4. TRANSFORMTION OF THE CONFLUENT MULTIPLE HYPERGEOMETRIC FUNCTION ${}^{(k)}_{(2)}\phi^{(n)}_{AC}$.

Since we can write

$${}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n;\, x_1, \ldots, x_n\right)$$

$$= \frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_{k+1}) \ldots \Gamma(b_n)}\left[1 - x_1E_1 - \ldots - x_nE_n\right]^{-a}\left\{\frac{\Gamma(b_{k+1}) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)}\right\}$$

$$= \frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_{k+1}) \ldots \Gamma(b_n)}\left[1 - x_{m+1} - \ldots - x_n\right]^{-a}$$

$$\left[1 - \frac{x_1E_1 + \ldots + x_mE_m + x_{m+1}\Delta_{m+1} + \ldots + x_n\Delta_n}{1 - x_{m+1} - \ldots - x_n}\right]^{-a}\left\{\frac{\Gamma(b_{k+1}) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)}\right\}$$

Therefore, we derive

$$(4.1)\quad {}^{(k)}_{(2)}\phi^{(n)}_{AC}\left(a, b_{k+1}, \ldots, b_n;\, c_1, \ldots, c_n;\, x_1, \ldots, x_n\right)$$

$$= \left(1 - x_{m+1} - \ldots - x_n\right)^{-a}\, {}^{(k)}_{(2)}\phi^{(n)}_{AC}\Big(a, b_{k+1}, \ldots, b_m, c_{m+1} - b_{m+1}, \ldots, c_n - b_n; c_1, \ldots, c_n;$$

$$\frac{x_1}{1-x_{m+1}-\ldots-x_n},\ldots,\frac{x_m}{1-x_{m+1}-\ldots-x_n},\frac{-x_{m+1}}{1-x_{m+1}-\ldots-x_n},\ldots,$$
$$\left.\frac{-x_n}{1-x_{m+1}-\ldots-x_n}\right),$$

which for $m = k$ gives (2.6); while for $k = 0$ and $m = k$, it reduces to (3.2).

Thus, by specializing the parameters and variables, we can derive different transformation formulas for hypergeometric functions of one, two, three and four variables. These transformation formulas may be useful in other topics of mathematical analysis.

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SOMEWHAT CONTINUOUS FUNCTIONS ON FUZZIFIED TOPOLOGICAL SPACE

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ABSTRACT : In this paper the concept of somewhat continuous functions, somewhat open functions are introduced in fuzzified topological space and some interesting properties of these functions are investigated.

Key words : Fuzzified topological space, somewhat continuous functions, somewhat open function.
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1. INTRODUCTION

The theory of fuzzy topological spaces was introduced and developed by C.L. Chang [2] and since then various notions in classical topology have been extended to fuzzy topological spaces. The concept of somewhat functions was introduced by Karl R. Genry and Hughes B. Hoyle [3] and this concept was studied in connection with the idea of feebly continuous function and feebly open functions introduced by Zdenek Frolik [4]. In [5] G. Thangaraj and G. Balasubranian introduced these concepts in fuzzy topological spaces. In this paper we have discussed the properties of somewhat continuous and somewhat open functions in fuzzified topological spaces.

2. PRELIMINARIES

In this section we recall some definitions and results that will be used in the sequel.

Definition 2.1[7] A fuzzified topology on a nonempty set X is fuzzy subset of power set of X, P(X) i.e., a function $\tau : P(X) \to [0, 1]$, satisfying the following conditions:

$\tau(X) = \tau(\phi) = 1$

$\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, A, $B \in P(X)$

$\tau(\cup A_i) \geq \underset{i}{\wedge} \tau(A_i)$ for any sub collection $\{A_i\}$ of P(X).

The pair (X, τ) will be called a fuzzified topological space (FTS). $\tau(A)$ is the degree of openness of A and $\tau(A^c)$ is the degree of closedness of A, where A^c is the complement of A.

Definition 2.2[7] A fuzzified co-topology on a nonempty set X is a fuzzy subset of P(X) i.e., a function $\omega : P(X) \rightarrow [0, 1]$, satisfying the following conditions:

$\omega(X) = \omega(\phi) = 1$

$\omega(A \cup B) \geq \omega(A) \wedge \omega(B)$, A, B $\in$ P(X)

$\tau(\cap A_i) \geq \underset{i}{\wedge} \tau(A_i)$ for any sub collection $\{A_i\}$ of P(X).

Proposition 2.3[7] Let τ be a fuzzified topology on X and $\omega_\tau : X \rightarrow [0, 1]$ be defined as $\omega_\tau(A) = \tau(A^c)$. Then ω_τ is a fuzzified co-topology on X.

Proposition 2.4[7] Let ω be a fuzzified co-topology on X and $\tau_\omega : X \rightarrow [0, 1]$ be defined as $\tau_\omega(A) = \omega(A^c)$. Then τ_ω is a fuzzified topology on X.

Proposition 2.5[7] If τ is a fuzzified topology and ω is a fuzzified co-topology on X then $\tau_{\omega\tau} = \tau$ and $\omega_{\tau\omega} = \omega$

Proposition 2.6[7] Let (X, τ) be a fuzzified topology and $Y \subseteq X$. Then $\tau_Y : P(Y) \rightarrow [0, 1]$ given by

$\tau_Y(U) = \vee\{\tau(V) : U = Y \cap V\}$ is a fuzified topology on Y.

τ_Y is then called fuzzified subspace topology.

Definition 2.7[7] Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be continuous with respect to τ and δ if $\tau(f^{-1}(U)) \geq \delta(U)$ for each $U \in P(Y)$.

Result 2.8. Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is continuous with respect to τ and δ iff $\omega_\tau(f^{-1}(U)) \geq \omega_\delta(U)$ for each $U \subseteq Y$.

Definition 2.9[7] Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be open with respect to τ and δ if $\tau(G) \leq \delta(f(G))$ for each $G \in P(X)$.

Definition 2.10[7] Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be closed with respect to τ and δ if $\omega_\tau(G) \leq \omega_\delta(f(G))$ for each $G \in P(X)$.

3. SOMEWHAT CONTINUOUS FUNCTIONS IN FUZZIIFIED TOPOLOGICAL SPACE

In this section we introduce and discuss the properties of somewhat continuous function in fuzzified topological spaces.

Definition 3.1. Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be somewhat continuous with respect to τ and δ if for each $U \subseteq Y$, with $f^{-1}(U) \neq \phi$ there exist $\phi \neq V \subseteq X$ such that $V \subseteq f^{-1}(U)$ and $\tau(V) \geq \delta(U)$.

It is clear from the definition that a continuous function is somewhat continuous. That the converse is not always true is evident from the following example.

Example 3.2. Let $X = \{a, b, c\}$, μ and ν are fuzzified topologies on X defined as :

	{a}	{b}	{c}	{a, b}	{a, c}	{b, c}	ϕ	X
μ	.7	.5	.1	.5	.3	.1	1	1
ν	.1	0	.1	.6	.1	0	1	1

Let $g : (X, \mu) \rightarrow (Y, \nu)$ be the identity function. Then $\mu[g^{-1}(\{a, b\})] = \mu(\{a, b\} = .5 < .6 = \nu(\{a, b\})$. Therefore g is not continuous. However there exists $\{a\} \subseteq \{a, b\}$ such that $\mu(\{a\}) > \nu(\{a, b\})$. Hence *g* is somewhat continuous.

Result 3.3. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is somewhat continuous with respect to τ and δ iff for each $U \subseteq Y$, with $f^{-1}(U) \neq X$ there exist $V \subsetneq X$ such that and $f^{-1}(U) \subseteq V$ and $\omega_\tau(V) \geq \omega_\delta(U)$.

Proof. Let $f : (X, \tau) \rightarrow (Y, \delta)$ be somewhat continuous. Consider $U \subseteq Y$ such that $f^{-1}(U) \neq X$. Then $U^c \subseteq Y$ such that $f^{-1}(U^c) \neq \phi$.

As f is somewhat continuous, there exists $\phi \neq W \subseteq X$:

$W \subseteq f^{-1}(U^c)$ and $\tau(W) \geq \delta(U^c) \Rightarrow W \subseteq \{f^{-1}(U)\}^c$ and $\omega_\tau(W^c) \geq \omega_\delta(U)$

$\Rightarrow f^{-1}(U) \subseteq W^c$ and $\omega_\tau(W^c) \geq \omega_\delta(U)$.

Let $V = W^c$, then $V \subsetneq X : f^{-1}(U) \subseteq V$ and $\omega_\tau(V) \geq \omega_\delta(U)$

Converse.

Let $U \subseteq Y$ such that $f^{-1}(U) \subseteq X$. Then $U^c \subseteq Y$ such that $f^{-1}(U^c) \neq \phi$. So by hypothesis there exist $V \subsetneq X : f^{-1}(U^c) \subseteq V$ and $\omega_\tau(V) \geq \omega_\delta(U^c)$ which gives $\{f^{-1}(U)\}^c \subseteq V$ and $\omega_\tau(V) \geq \omega_\delta(U^c) \Rightarrow V^c \subseteq f^{-1}(U)$ and $\tau(V^c) \geq \delta(U)$. Let $V^c = W$.

Then $W \neq \phi$ and $W \subseteq f^{-1}(U)$ and $\tau(W) \geq \delta(U)$.

Result 3.4. Let (X, τ), (Y, δ) and (Z, η) be FTSs.

Let $f : (X, \tau) \rightarrow (Y, \delta)$ and $g : (Y, \delta) \rightarrow (Z, \eta)$ are somewhat continuous funcitons, then $go f : (X, \tau) \rightarrow (Z, \eta)$ is continuous, if $f^{-1}(V) \neq \phi$ for all $\phi \neq V \subseteq Y$.

Proof. Let $U \subseteq Z$ such the $(gof)^{-1}(U) \neq \phi$.

We have $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \neq \phi$ implies $g^{-1}(U) \neq \phi$. As g is somewhat continuous, there exists $\phi \neq W \subseteq Y : W \subseteq g^{-1}(U)$ and $\delta(W) \geq \eta(U)$.

Again by hypothesis $\phi \neq W \Rightarrow f^{-1}(W) \neq \phi$. As f is somewhat continuous, there exists $\phi \neq V \subseteq X : V \subseteq f^{-1}(W)$ and $\tau(V) \geq \delta(W)$. Thus $V \subseteq f^{-1}(g^{-1}(U))$ and $\tau(V) \geq \delta(W) \geq \eta(U)$.

That is, there exists $\phi \neq V \subseteq X : V \subseteq (gof)^{-1}(U))$ and $\tau(V) \geq \eta(U)$.

The following example shows that the condition $f^{-1}(V) \neq \phi$ for all $\phi \neq V \subseteq Y$ is sufficient but not necessary.

Example 3.5. Let X = {a, b, c}, τ, μ and ν are fuzzified topologies on X defined as :

	{a}	{b}	{c}	{a, b}	{a, c}	{b, c}	ϕ	X
μ	.7	.5	.1	.5	.3	.1	1	1
ν	.1	0	.0	.6	.1	0	1	1
τ	.7	.4	0	.4	0	.1	1	1

Let $f : (X, \mu) \to (X, \tau)$ be defined as $f(a) = a$, $f(b) = f(c) = c$ and $g : (X, \tau) \to (X, \nu)$ be the identity function.

Then it is easy to verify that f, g and gof are somewhat continuous though $f^{-1}(\{b\}) = \phi$.

Definition 3.6. Let (X, τ) be a FTS. A subset $D \subseteq X$ is said to be dense in X if there exists no subset $V \subsetneq X$ such that $D \subseteq V$ and $\omega_\tau(V) > \omega_\tau(D)$.

Result 3.7. Image of a dense subset under a somewhat continuous and closed function is dense.

Proof. $f : (X, \tau) \to (Y, \mu)$ be somewhat continuous and D be dense in X. Consider $f(D)$. If $f(D) = X$, we are done. So let $f(D) \neq X$. Suppose $f(D)$ is not dense, then there exists $B \subsetneq Y : f(D) \subsetneq B$ and $\omega_\mu(B) > \omega_\mu(f(D))$. Since f is somewhat continuous, for $B \subsetneq Y$ there exists $F \subsetneq X : f^{-1}(B) \subseteq F$ and $\omega_\tau(F) \geq \omega_\mu(B)$. Again $f(D) \subseteq B \Rightarrow D \subseteq f^{-1}(B) \subseteq F \Rightarrow D \subseteq F$. Also, $\omega_\tau(F) \geq \omega_\mu(B) > \omega_\mu(f(D)) \geq \omega_\tau(D)$, since f is closed. But this is a contradiction to the fact that D is dense in X.

The following example shows that the above result need not be true if the condition of closed of f is omitted.

Example 3.8. Let X = {a, b, c, d} and τ and μ be fuzzified topologies on X defined as:

$\tau(A) = 1$, $A = X, \phi$

$= .7$, $A = \{c, d\}$

$= .5$, otherwise.

$\mu(A) = 1$, $A = X, \phi$

$= .6$, $A = \{b\}$

$= .4$, otherwise.

Let $f : (X, \tau) \to (Y, \mu)$ be given by $f(a) = b$, $f(b) = c$, $f(c) = d$, $f(d) = a$.

Then f is somewhat continuous and $D = \{a, b\}$ is dense in X.

But $f(D) = \{b, c\}$ is not dense in Y, as there exists $\{a, c, d\} \supseteq \{b, c\}$ such that $\omega_\mu(\{a, c, d\}) = \mu(\{b\}) = .6 > .4 = \mu(\{a, d\}) = \omega_\mu(\{b, c\}) = \omega_\mu(f(D))$.

Note that here f is not closed.

Result 3.9. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat continuous. Then for any $A \subseteq X$,

$f/_A : (A, \tau_A) \to (Y, \delta)$ is somewhat continuous.

Proof. Since $f/_A$ is the composition of the inclusion map $j : (A, \tau_A) \to (X, \tau)$ and $f : (X, \tau) \to (Y, \delta)$, the result follows from Result 3.4.

Result 3.10. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat continuous where Y is a subspace of Z. Then $h : X \to Z$, where h is obtained by expanding the range of f is somewhat continuous.

Proof. Let μ denote the fuzzified topology on Z. Then $\delta = \mu_Y$. Let $U \subseteq Z$ with $h^{-1}(U) \neq \phi$. We have $V = Y \cap U \subseteq U$. Then $f^{-1}(V) = h^{-1}(V) = h^{-1}(Y \cap U) = h^{-1}(Y) \cap h^{-1}(U) = h^{-1}(U)$. So $f^{-1}(V) \neq \phi$. Since f is somewhat continuous, there exists $\phi \neq W \subseteq X : W \subseteq f^{-1}(V)$ and $\tau(W) \geq \delta(V) = \mu_Y(V) \geq \mu(V)$. Thus $W \subseteq h^{-1}(U)$ and $\tau(W) \geq \mu(V)$.

Result 3.11. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat continuous and $f(X) \subseteq Z \subseteq Y$.

Then $g : (X, \tau) \to (Z, \delta_z)$ is somewhat continuous.

Proof. Let $U \subseteq Z$ with $g^{-1}(U) \neq \phi$.

As $f(X) \subseteq Z$, then for any $V \subseteq Y : U = Z \cap V$, $f^{-1}(V) = g^{-1}(U)$.

As f is somewhat continuous, there exists $\phi \neq W \subseteq X : W \subseteq f^{-1}(V)$ and $\tau(W) \geq \delta(V)$.

But this is true for any $V \subseteq Y$ satisfying $U = Z \cap V$.

So $\tau(W) \geq \wedge\{\delta(V) : U = Z \cap V\} = \delta_z(U)$.

Hence there exists $\phi \neq W \subseteq X : W \subseteq g^{-1}(V)$ and $\tau(W) \geq \delta_z(U)$.

Result 3.12. Let $f : (X, \tau) \to (Y, \delta)$ and suppose A and B be subsets of X :

$X = A \cup B$ and $\tau(A) = 1 = \tau(B)$. If $f_A : (A, \tau_A) \to (Y, \delta)$ and $f_B : (B, \tau_B) \to (Y, \delta)$ are somewhat continuous then f is somewhat continuous.

Proof. Let $U \subseteq Y$ and $f^{-1}(U) \neq \phi$. As $f^{-1}(U) = f_A^{-1}(U) \cup f_B^{-1}(U)$, either $f_A^{-1}(U) \neq \phi$ or $f_B^{-1}(U) \neq \phi$. Suppose $f_A^{-1}(U) \neq \phi$. As f_A is somewhat continuous, there exists $\phi \neq W \subseteq A : W \subseteq f_A^{-1}(U)$ and $\tau_A(W) \geq \delta(U)$. But $\tau(A) = 1$ implies $\tau_A(W) = \tau(W)$.

Thus we have $W \subseteq f_A^{-1}(U) \subseteq f^{-1}(U)$ and $\tau(W) \geq \delta(U)$.

Definition 3.13. Let τ and δ be two fuzzified topologies on a set X. τ is said to be weakly equivalent to δ if for any $\phi \neq W \subseteq X$, there exists $\phi \neq U, V \subseteq W$ such that $\tau(U) \geq \delta(W)$ and $\delta(V) \geq \tau(W)$.

Remark 3.14. τ and δ are weakly equivalent on X iff the identity function from (X, τ) onto (X, δ) is somewhat continuous in both directions.

Proof. The proof is straightforward.

Result 3.15. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat continuous. If μ is weakly equivalent to τ, then $f : (X, \mu) \to (Y, \delta)$ is somewhat continuous.

Proof. Let $U \subseteq Y$ and $f^{-1}(U) \neq \phi$. As f is somewhat continuous, there exists $\phi \neq V \subseteq X$: $V \subseteq f^{-1}(U)$ and $\tau(V) \geq \delta(U)$.

Again μ is weakly equivalent to δ, therefore there exists $\phi \neq W \subseteq V : \mu(W) \geq \tau(V)$.

Result 3.16. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat continuous and onto. If δ is weakly equivalent to ν, then $f : (X, \tau) \to (Y, \nu)$ is somewhat continuous.

Proof. Let $U \subseteq Y$ such that $f^{-1}(U) \neq \phi$. Then $U \neq \phi$. Again δ is weakly equivalent to ν implies there exists $\phi \neq W \subseteq U : \delta(W) \geq \nu(U)$. As f is onto $W \neq \phi$ implies $f^{-1}(W) \neq \phi$.

Now $f : (X, \tau) \to (Y, \delta)$ is somewhat continuous, therefore, there exists $\phi \neq V \subseteq X$ such that $V \subseteq f^{-1}(W)$ and $\tau(V) \geq \delta(W)$.

Thus there exists $\phi \neq V \subseteq X$ such that $V \subseteq f^{-1}(W)$ and $\tau(V) \geq \nu(U)$.

Combining Result 3.15 and 3.16 we have

Result 3.17. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat continuous and onto. If μ is weakly equivalent to τ and ν is weakly equivalent to δ, then $f : (X, \mu) \to (Y, \nu)$ is somewhat continuous.

Definition 3.18. A fuzzified topological space is said to be separable if there exist a countable dense subset.

Result 3.19. Let $f : (X, \tau) \rightarrow (Y, \delta)$ be closed and somewhat continuous. If X is separable then so is Y.

Proof. Since image of a countable set is countable, the result follows from Result 3.7.

4. SOMEWHAT OPEN FUNCTIONS IN FUZZIFIED TOPOLOGICAL SPACE

In this section we introduce and discuss the properties of somewhat open function in fuzzified topological spaces.

Definition 4.1. Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be somewhat open with respect to τ and δ if for each $\phi \neq U \subseteq X$, there exist $\phi \neq V \subseteq X : V \subseteq f(U)$ and $\delta(V) \geq \tau(U)$.

It is clear from the definition that an open function is somewhat open. That the converse is not always true is evident from the following example.

Example 4.2. Let $X = \{a, b, c\}$, τ and μ are fuzzified topologies on X defined as :

	{a}	{b}	{c}	{a, b}	{a, c}	{b, c}	ϕ	X
τ	.1	.1	.1	.5	.1	.1	1	1
μ	.6	.2	.2	.2	.2	.2	1	1

Let $f : (X, \tau) \rightarrow (Y, \mu)$ be the identity function.

Then f is not open, since $f(\{a, b\}) = \{a, b\}$ and $\mu[f(\{a, b\})] = .2 < .5 = \tau\{a, b\}$.

However f is somewhat open as there exists $\{a\} \subseteq f(\{a, b\})$ such that

$\mu(\{a\}) = .6 > .5 = \tau\{a, b\}$

Definition 4.3. Let (X, τ) and (Y, δ) be two FTSs. A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be somewhat closed with respect to τ and δ if for each $U \not\subseteq X$, there exist $V \not\subseteq Y$ such that $f(U) \subseteq V$ and $\omega_\delta(V) \geq \omega_\tau(U)$.

Result 4.4. Let $f: (X, \tau) \rightarrow (Y, \mu)$ is somewhat continuous, and $g : (Y, \mu) \rightarrow (Z, \nu)$ is somewhat continuous, the $(gof) : (X, \tau) \rightarrow (Z, \nu)$.

Proof. Let $\phi \neq U \subseteq X$. Consider $(gof)(U)$. Since $f : (X, \tau) \to (Y, \mu)$ is somewhat continuous, there exist $\phi \neq V \subseteq Y : V \subseteq f(U)$ and $\mu(V) \geq \tau(U)$. Again $g : (Y, \mu) \to (Z, v)$ is somewhat continuous, so there exists $\phi \neq W \subseteq Z : W \subseteq g(V)$ and $v(W) \geq \mu(V)$.

Thus we have $\phi \neq W \subseteq g(V) \subseteq g(f(U)) = (gof)(U)$ such that $v(W) \geq \tau(U)$.

Result 4.5. Inverse image of a dense subset under a somewhat closed and continuous function is dense.

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat closed and continuous function. Let D be dense in Y. consider $f^{-1}(D)$. Suppose $f^{-1}(D)$ is not dense in X. Then there exists $F \subsetneq X : f^{-1}(D) \subseteq F$ and $\omega_\tau(F) > \omega_\tau(f^{-1}(D))$. Now $F \subsetneq X \Rightarrow f(F) \subsetneq Y$. As f is somewhat closed, there exists $B \subsetneq Y : f(F) \subseteq B$ and $\omega_\delta(B) \geq \omega_\tau(F)$.

Thus $D \subseteq f(F) \subseteq B$ and $\omega_\delta(B) \geq \omega_\tau(F) > \omega_\tau(f^{-1}(D)) \geq \omega_\delta(D)$, since f is continuous. But this is contradiction to that fact that D is dense in Y. Hence $f^{-1}(D)$ is dense in X.

Result 4.6. Let $f : (X, \tau) \to (Y, \delta)$ be one-one and onto. Then f is somewhat open iff f is somewhat colsed.

Proof. Suppose f is open. Let $U \subseteq X : f(U) \neq Y$. To find $V \subsetneq Y$ such that $f(U) \subseteq V$ and $\omega_\delta(V) \geq \omega_\tau(U)$. If $U = \phi$, then $f(U) = \phi$. We choose $V = \phi$.

So let $\phi \neq U \subsetneq X$. Then $\phi \neq f(U) \subsetneq Y$. Let $U^c = W$, then $\phi \neq W \subsetneq X$. Since f is somewhat open, there exists $\phi \neq V \subseteq Y$:

$V \subseteq f(W)$ and $\tau(W) \geq \delta(V) \Rightarrow f(W)^c \subseteq V^c$ and $\omega_\tau(W^c) \geq \omega_\delta(V^c)$

$\Rightarrow f(W^c) \subseteq V^c$ and $\omega_\tau(U) \geq \omega_\delta(V^c)$. Let $V^c = F$. Since $\phi \neq V$, $Y \neq V^c = F$. Thus for $U \subseteq X : f(U) \subsetneq Y$, there exists $F \subsetneq Y : f(U) \subseteq F$ and $\omega_\tau(U) \geq \omega_\delta(F)$. Hence f is somewhat closed.

Converse. Let $\phi \neq U \subseteq X$. If $U = X$, $f(U) = Y$. Choosing $V = Y$, we are done.

So let $\phi \neq U \subsetneq X$. Let $G = U^c$, then $\phi \neq G \subsetneq X$ and so $f(G) \subsetneq Y$, as f is onto.

Hence by hypothesis there exists $W \subsetneq Y$:

$f(G) \subseteq W$ and $\omega_\delta(W) \geq \omega_\tau(G) \Rightarrow W^c \subseteq f(G)^c$ and $\delta(W^c) \geq \tau(G^c)$

$\Rightarrow W^c \subseteq f(G^c)$ and $\delta(W^c) \geq \tau(G^c) \Rightarrow W^c \subseteq f(U)$ and $\delta(W^c) \geq \tau(U)$

Let $W^c = V$. Then $\phi \neq V \subseteq Y : V \subseteq f(U)$ and $\delta(V) \geq \tau(U)$. Hence f is somewhat open.

Result 4.7. Let $f : (X, \tau) \to (Y, \delta)$ and suppose A and B be subsets of X:

$X = A \cup B$. If $f : (A, \tau_A) \to (Y, \delta)$ and $f : (B, \tau_B) \to (Y, \delta)$ are somewhat open then f is somewhat open.

Proof. Let $\phi \neq U \subseteq X$. If $U \cap B = \phi$, then $f(U) = f_A(U)$.

Since f_A is somewhat open, there exists $\phi \neq V \subseteq Y : V \subseteq f_A(U)$ and $\delta(V) \geq \tau_A(U)$. But $\tau_A(U) \geq \tau(U)$. Therefore $\phi \neq V \subseteq Y : V \subseteq f(U)$ and $\delta(V) \geq \tau(U)$.

Similar is the case if $U \cap A = \phi$. So let $U \cap A \neq \phi \neq U \cap B$. Then $f(U) = f_A(U) \cup f_B(U)$.

As f_A is somewhat open, there existss $\phi \neq V \subseteq Y : V \subseteq f_A(U)$ and $\delta(V) \geq \tau_A(U) \geq \tau(U)$.

As f_B is somewhat open, there exists $\phi \neq W \subseteq Y : W \subseteq f_B(U)$ and $\delta(W) \geq \tau_A(U) \geq \tau(U)$.

Then there exists $\phi \neq F = V \cap W \subseteq Y$:

$F \subseteq f_A(U) \cup f_B(U) = f(U)$ and $\delta(F) = \delta(V \cap W) \geq \delta(V) \wedge \delta(W) \geq \tau(U)$.

Remark 4.8. τ and δ are weakly equivalent on X iff the identity function from (X, τ) onto (X, δ) is somewhat open in both directions.

Result 4.9. Let $f : (X, \tau) \to (Y, \delta)$ be somewhat open funciton. If μ is weakly equivalent to τ and ν is weakly equivalent to τ, then $f : (X, \mu) \to (Y, \nu)$ is somewhat open function.

Proof. The proof is straightforward.

Definition 4.10. A $f : (X, \tau) \to (Y, \delta)$ is a said to be somewhat homeomorphism if f is one-one, onto, somewhat open and somewhat continuous.

Remark 4.11. If f is a somewhat homeomorphism then f^{-1} is also a somewhat homeomorphism.

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ON METRIC DENSITY AS A SET FUNCTION

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ABSTRACT : This paper endeavours to reveal the topological character of the subclasses $\mathcal{D}(x)$ and $\mathcal{M}(x)$ (in the class of all Lebesgue measurable subsets of $\mathbb{R}$) defined by Martin in his paper [3] and shows that in this respect, they are similar to their duals $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ which are the collection of all points at which E has a density and that subcollection of it where this value of density is either 'zero' or 'one'.

Key words : Metric (Lebesgue) density, Borel, $F_{\sigma\delta}$, $\overline{D}_x$-measurable, finitely additive outer measure.
AMS Subject Classification : 28-A (2000).

1. INTRODUCTION

Given a Lebesgue measurable set E ($\subseteq \mathbb{R}$) and x ($\in \mathbb{R}$), quantities such as $\overline{D}_x(E) = \limsup_{I\to x}\frac{\mu(E\cap I)}{\mu(I)} = \sup_{\{I_k\}}\left\{\limsup_{k\to\infty}\frac{\mu(E\cap I_k)}{\mu(I_k)}; I_k \to x\right\}$ and $\underline{D}_x(E) = \liminf_{I\to x}\frac{\mu(E\cap I)}{\mu(I)} = \inf_{\{I_k\}}\left\{\liminf_{k\to\infty}\frac{\mu(E\cap I_k)}{\mu(I_k)}; I_k \to x\right\}$ (where each I_k is a non-degenerate interval of $\mathbb{R}$ and $I_k \to x$ means that x ($\in I_k$) for each k ($\in \mathbb{N}$) and $\mu(I_k) \to 0$, μ being the Lebesgue measure in $\mathbb{R}$) are called respectively as the upper and lower metric (or Lebesgue) densities of E at x. In particular if, $\overline{D}_x(E) = \underline{D}_x(E)$, E is said to possess a density at the point x which is the common value of the above two quantitites and is denoted by $D_x(E)$.

By virtue of the Density theorem of Lebesgue, given any linear measurable set E,

$$\begin{aligned} D_x(E) &= 1, \quad \mu \text{ - almost everywhere on } E \\ &= 0, \quad \mu \text{ - almost everywhere on } \mathbb{R}\setminus E. \end{aligned}$$

Thus the concept of density as defined by Lebesgue and put to use in his famous theorem is usually considered as a point function where the set is kept fixed and the point is made to vary.

Thus given any linear measurable set E, as per the density theorem stated above, we can define a point function

$\overline{D}_E : \mathbb{R} \to [0, 1]$ such that -

$$\begin{aligned}\overline{D}_E(x) &= D_x(E) = 1, \ \mu \text{ - almost everywhere on } E\\ &= 0, \ \mu \text{ - almost everywhere on } \mathbb{R}\backslash E\end{aligned}$$

which accordingly gives rise to the following two sets

$\mathcal{D}(E) = \{x \in \mathbb{R} \mid D_x(E) \text{ exists}\}$

and $\mathcal{D}^*(E) = \{x \in \mathbb{R} \mid D_x(E) = 0 \text{ or } 1\}$

But this concept of density may also be viewed in an entirely different context, where in place of the set, the point is fixed and a function.

$\overline{D}_x : \mathcal{M} \to [0, 1]$ is defined on the class $\mathcal{M}$ of all Lebesgue measurable sets of $\mathbb{R}$ by means of the formula–

$$\overline{D}_x(E) = \overline{D}_E(x) \text{ for every } E \ (\in \mathcal{M})$$

Martin [2] found that the set function $\overline{D}_x$ is a finitely sub-additive outer measure function on the class $\mathcal{M}$ of Lebesgue measurable sets and more interestingly the class $\mathcal{M}(x)$ of $\overline{D}_x$ - measurable sets can be given the following precise formulation–

$$\mathcal{M}(x) = \{E \in \mathcal{D}(x);\ D_x(E) = 0 \text{ or } 1\} \quad (*)$$

Now if $\mathcal{D}(x)$ stands for the class of all measurable sets for which $D_x(E)$ exists, then the set $\mathcal{D}(E)$ and the class $\mathcal{D}(x)$ can be considered as dual of each other (with the role of set and point interchanged) and a similar relation holds between $\mathcal{D}^*(E)$ and $\mathcal{M}(x)$. Moreover, viewed in this context, the assertion (*) may be regarded as the dual formulation of the density theorem stated above.

Here in this paper, we endeavour to unravel an interesting similarity in the topological character of the two structures $\mathcal{D}(E)$ and $\mathcal{D}(x)$ (and also of $\mathcal{D}^*(E)$ and $\mathcal{M}(x)$) which according to the above mode of representing their definitions are dual of each other.

It is already known that in the topology of the real line derived from the usual metric structure in it, the sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ are Borel $F_{\sigma\delta}$ sets. Results of this type follow as immediate consequences of more general theorems (related to interval functions) proved by the present author in [1]. There it was proved that for any Lebesgue measurable set E (of

$\mathbb{R}^n$), both the sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ are of the Borel $F_{\sigma\delta}$ type. It is interesting to note from the deductions given in the following paragraphs that their dual structures viz. $\mathcal{D}(x)$ and $\mathcal{M}(x)$ (with the role of set and point interchanged) also bear the same property of being Borel sets of the same Borel type in the topology of measurable subsets of the real line derived from the pseudo-metric $\sigma . \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ defined by —

$$\sigma(E, F) = \mu(E \,\Delta\, F) \text{ where } E, F\ (\in \mathcal{M}).$$

Main Results. The results here are derived as particular cases of more general theorems concerning Lebesgue integrable functions on $\mathbb{R}$, the class being denoted by the symbol $\mathcal{L}$. For any x ($\in \mathbb{R}$) chosen and fixed we define—

$$\Phi_x(f) = \limsup_{I\to x} \frac{\int_I f d\mu}{\mu(I)} \qquad (f \in \mathcal{L})$$

$$\Psi_x(f) = \liminf_{I\to x} \frac{\int_I f d\mu}{\mu(I)}$$

which acording to their definition are both real-valued functions on $\mathcal{L}$. We also set,

$$A(x) = \left\{ f \in \mathcal{L}\colon \lim_{I\to x} \frac{\int_I f d\mu}{\mu(I)} \text{ exists} \right\},$$

$$\text{and } \mathcal{L}(x) = \left\{ f \in \mathcal{L}\colon \limsup_{I\to x} \frac{\int_I f d\mu}{\mu(I)} = 0 \text{ or } 1 \right\}$$

It is well-known that the class $\mathcal{L}$ is a metric space with respect to the pseudo-metric $\rho(f, g) = \int_R |f - g| d\mu$.

The following proposition illustrates two essential identities needed to proceed further and which are not hard to justify.

Proposition. If for any n ($\in \mathbb{N}$), $\mathcal{J}_x^{(n)} = \left\{ I;\ x \in I \text{ and } \frac{1}{n+1} < \mu(I) \leq \frac{1}{n} \right\}$, then $\Phi_x = \limsup_{n\to\infty} \Phi_x^{(n)}$ and $\Psi_x = \liminf_{n\to\infty} \Psi_x^{(n)}$, where

$$\Phi_x^{(n)}(f) = \sup\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} \text{ and}$$

$$\Psi_x^{(n)}(f) = \inf\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}.$$

Theorem 1. $A(x)$ is a Borel $F_{\sigma\delta}$ subset of $(\mathcal{L}, \rho)$.

Lemma 1.1. For any n ($\in \mathbb{N}$), both $\Phi_x^{(n)}$ and $\Psi_x^{(n)}$ are continuous functions or $\mathcal{L}$.

Proof of the Lemma. For any f, g ($\in \mathcal{L}$).

$$\begin{aligned}\left|\Phi_x^{(n)}(f) - \Phi_x^{(n)}(g)\right| &= \left|\sup\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} - \sup\left\{\frac{\int_I g d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}\right| \\ &\leqq \sup\left\{\frac{\int_I |f-g|}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} \\ &\leqq n.\int_R |f-g| d\mu \ldots\end{aligned}$$

and this inequality is enough to exhibit the continuity of $\Phi_x^{(n)}$.

$$\text{Also, } \left|\Psi_x^{(n)}(f) - \Psi_x^{(n)}(g)\right| = \left|\inf\left\{\frac{\int_I f d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} - \inf\left\{\frac{\int_I g d\mu}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\}\right|$$

$$\leqq \sup\left\{\frac{\int_I |f-g|}{\mu(I)}; I \in \mathcal{J}_x^{(n)}\right\} \leqq n\int_R |f-g| d\mu$$

which similarly shows that the function $\Psi_x^{(n)}$ is continuous.

Lemma 1.2. For any r ($\in$ R), the sets $\{\Phi_x \geqq r\}$ and $\{\Psi_x \leqq r\}$ are G_δ in $(\mathcal{L}, \rho)$.

Proof. By virtue of the proposition stated in the beginning of this section,

$$\{f \in \mathcal{L};\ \Phi_x(f) \geqq r\} = \bigcap_p \bigcap_n \bigcup_{k \geqq n} \left\{\Phi_x^{(k)}(f) > r - \frac{1}{p}\right\}$$

and $$\{f \in \mathcal{L};\ \Psi_x(f) \leqq r\} = \bigcap_p \bigcap_n \bigcup_{k \geqq n} \left\{\Psi_x^{(k)}(f) < r + \frac{1}{p}\right\}.$$

Now using Lemma 1.1, the result follows.

Proof of the theorem. We may write,

$$A_x = \mathcal{L} \setminus \{f \in \mathcal{L};\ \Psi_x(f) < \Phi_x(f)\} = \mathcal{L} \setminus \bigcup_k \{f \in \mathcal{L};\ \Psi_x(f) < r_k < \Phi_x(f)\}$$

where $\{r_k;\ k \in \mathbb{N}\}$ is the set of all rationals in $\mathbb{R}$. But for any k, the set

$$\{f \in \mathcal{L};\ \Psi_x(f) < r_k < \Phi_x(f)\} = \{f \in \mathcal{L};\ \Psi_x(f) < r_k\} \cap \{f \in \mathcal{L};\ \Phi_x(f) > r_k\}$$

$$= \bigcup_q \bigcup_q \left[\left\{f \in \mathcal{L}; \Psi_x(f) \leqq r_k - \frac{1}{p}\right\} \bigcap \left\{f \in \mathcal{L}; \Phi_x(f) \geqq k + \frac{1}{q}\right\}\right]$$

is $G_{\delta\sigma}$ by Lemma 1.2.

Hence the theorem.

For any n ($\in \mathbb{N}$), we denote by the symbol $J_x^{(n)}$ the interval $\left(\dot{x} - \frac{1}{n}, x + \frac{1}{n}\right)$ and write

$$\eta_x(f) = \lim_{I \to x} \frac{\int_I f d\mu}{\mu(I)} \quad \text{and} \quad \eta_x^{(n)}(f) = \lim_{I \to x} \frac{\int_{J_x^{(n)}} f d\mu}{\mu\left(J_x^{(n)}\right)}.$$

Theorem 2. $\mathcal{L}(x)$ is a Borel $F_{\sigma\delta}$ subset of $\mathcal{L}$.

Lemma 2.1. The function $\eta_x^{(n)}$ is continuous on $A(x)$.

Proof. Let f, g ($\in A(x)$). Then

$$\left|\eta_x^{(n)}(f)-\eta_x^{(n)}(g)\right| = \left|\frac{\int\limits_{J_x^{(n)}} f d\mu}{\mu\left(J_x^{(n)}\right)} - \frac{\int\limits_{J_x^{(n)}} g d\mu}{\mu\left(J_x^{(n)}\right)}\right|$$

$$= \frac{n}{2}\left|\int\limits_{J_x^{(n)}} f d\mu - \int\limits_{J_x^{(n)}} g d\mu\right| \leqq \frac{n}{2}\int\limits_{R} |f-g| d\mu$$

which finally settles the lemma.

Lemma 2.2. For any r ($\in \mathbb{R}$), the set $\{f \in \mathcal{L}, \eta_x(f) = r\}$ is $F_{\sigma\delta}$ in $A(x)$.

Proof.

$$\{f \in A(x);\ \eta_x(f) = r\} =$$

$$\bigcap_q \left[\left\{f \in A(x); \eta_x(f) < r + \frac{1}{q}\right\} \bigcap \left\{f \in A(x); \eta_x(f) > r - \frac{1}{q}\right\}\right]$$

But $\eta_x = \lim\limits_{n\to\infty} \eta_x^{(n)}$ and therefore,

$$\left\{f \in A(x); \eta_x(f) > r - \frac{1}{q}\right\} = \bigcup_k \bigcup_p \zeta_{k,p}^{(q)}(x), \text{ where}$$

$$\zeta_{k,p}^{(q)}(x) = \bigcap_{n \geq p} U_{k,n}^{(q)}(x) \text{ and } U_{k,n}^{(q)}(x) = \left\{f \in A(x);\ \eta_x^{(n)}(f) \geqq r - \frac{1}{q} + \frac{1}{k}\right\}$$

So by lemma 2.1., $\left\{f \in A(x); \eta_x(f) > r - \frac{1}{q}\right\}$ and similarly $\left\{f \in A(x); \eta_x(f) < r + \frac{1}{q}\right\}$ are F_σ and therefore the above set is an $F_{\sigma\delta}$ subset of $A(x)$.

Proof of the theorem. Since $\mathcal{L}(x) = \{f \in \mathcal{L};\ \eta_x(f) = 0 \text{ or } 1\}$ and $A(x)$ being an $F_{\sigma\delta}$ subset of $\mathcal{L}$ (by theorem 1), the theorem follows.

Thus we have proved in terms of Theorem 1 and 2 that for any fixed x in $\mathbb{R}$, the subclasses $A(x)$ and $\mathcal{L}(x)$ of the class $\mathcal{L}$ of Lebesgue integrable functions on $\mathbb{R}$ are Borel $F_{\sigma\delta}$ subsets of it. Had we chose instead of $\mathcal{L}$, the class $\mathcal{M}$ of Lebesgue measurable subsets of $\mathbb{R}$ to start with (which means that subclass of $\mathcal{L}$ consisting precisely of the characteristics functions of measurable sets) we would have reached as a consequence the following two conclusions–

Corollary 1. $\mathcal{D}(x)$ is a Borel $F_{\alpha\delta}$ subset of $(\mathcal{M}, \sigma)$

and

Corollary 2. $\mathcal{M}(x)$ is a Borel $F_{\sigma\delta}$ subset of (M, σ).

This shows that just as the sets $\mathcal{D}(E)$ and $\mathcal{D}^*(E)$ (for a fixed Lebesgue measurable set E) are Borel subsets of $\mathbb{R}$ (with its usual topology), so are their duals $\mathcal{D}(x)$ and $\mathcal{M}(x)$ (when the roles of the set and point are interchanged) having the same Borel type, in the topology of Lebesgue measurable subsets of $\mathbb{R}$.

Note. Using translation invariance of Lebesgue measure, it may be easily seen that the metric spaces $(\mathcal{D}(x), \sigma)$ and $(\mathcal{D}(y), \sigma)$ are isometric and therefore homeomorphic. The same reason suffices to show that $\mathcal{M}(x)$ and $\mathcal{M}(y)$ are also homeomorphic. Now if we write $\mathcal{M}_0(x) = \{E \in \mathcal{D}(x) : D_x(E) = 0\}$ and $\mathcal{M}_1(x) = \{E \in \mathcal{D}(x) : D_x(E) = 1\}$, then also $\mathcal{M}_0(x)$ and $\mathcal{M}_0(y)$ are isometric and therefore homeomorphic as Lebesgue measure is translation invariant and so also are $\mathcal{M}_0(x)$ and $\mathcal{M}_1(y)$ because of this fact and also that $\sigma(E\Delta F) = \sigma((\mathbb{R}\backslash E)\Delta(\mathbb{R}\backslash F))$ for any two $E, F(\in \mathcal{M})$. But $\mathcal{D}(x)$ and $\mathcal{M}(y)$ are not homeomorphic as it can be shown by virtue of a more general deduction [2] that both $\mathcal{M}_0(x)$ and $\mathcal{M}_1(x)$ are dense in $(\mathcal{M}, \sigma)$.

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ON SOME CONGRUENCES IN SEMIRING

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ABSTRACT : The concept of p-ideals in a semiring R was first introduced by Mukhopadhyay and Ghosh [4,5] and subsequently various characteristic features of a p-ideal in different classes of semirings were obtained in [4], [5], [6]. In the present paper, a congruence p_I induced by an ideal I of R is provided and certain basic results corresponding to this concept including the nature of the congruence when I is a p-ideal in particular, are established. In the last section, yet another congruence, Bourne p-congruence, compatible with the concept of p-ideals in a semiring is established and studied at length.

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1. INTRODUCTION AND PRELIMINARIES

A semiring is a non-empty set S together with two binary operations, called "addition", +, and "multiplication" (usually denoted by juxtaposition), such that S is multiplicatively a semigroup and additively a commutative semigroup and the multiplication is distributed accross the addition both from the left and from the right. a semiring is said to be *commutative* if it is multiplicatively commutative. An additively cancellative semiring is called a *halfring*. An *inversive semiring* [7] S is a semiring in which $(S, +)$ is an inverse semigroup, i.e., for each $a \in S$, there is a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. It is well known [3] that, in an inversive semiring S, we have $(ab)' = a'b = ab'$ and $(a + b)' = a' + b'$. $E^+(S)$ denotes the set of all additive idempotents of S. If an ideal of the semiring contains $E^+(S)$ then it is called a *full* ideal. A semiring S ia said to have an identity, if there exists $1_s \in S$ such that, $1_s x = x 1_s = x$ for each $x \in S$. The zero element of S, denoted by 0, is called an *absorbing zero* if $a0 = 0a = 0$ for all $a \in S$. A *k-ideal* [2] I of a semiring S is an ideal of S such that, if $a \in I$, $x \in S$ and $a + x \in I$, then $x \in I$. A *congruence* ρ on S is an equivalence relation on S such that $a\ \rho\ b$ in S implies $(a + c)\ \rho\ (b + c)$, $ac\ \rho\ bc$ and $ca\ \rho\ cb$ hold for all $a, b, c \in S$. The *congruence class* of an element $a \in S$ under ρ is denoted by $a\rho$. The *quotient semiring* $\{a\rho : a \in S\}$ of S under ρ is defined usually and is denoted by S/ρ. It is interesting to point out that, unlike rings, ideals and congruences, in general, do not correspond bijectively in semirings. Throughout this paper $\mathbb{N}$ represents the set of natural numbers.

Mukhopadhyay and Ghosh [4] introduced the idea of p-ideals in semirings. We recall the following definitions and results that will be useful in the sequel.

Definition 1.1. [4,5] An ideal I of a semiring S is called a *p-ideal* if for some $x \in S$, $n \in \mathbb{N}$,

$$nx + a = (n + 1)x \text{ and } a \in I \text{ implies } x \in I.$$

In particular, if S is *inversive,* then the definition boils down to the following:

if for some $x \in S$, $a + x = 2x$, $a \in I$ then $x \in I$.

Observe that all p-ideals are not k-ideals, as the ideal $I = 3Z_0^+ \setminus \{3\}$ is not a k-ideal, in the halfring Z_0^+ of all positive integers with zero but it is a p-ideal, as all ideals of a halfring are p-ideals. We also note that k-ideals are not p-ideals in general. Indeed, in the semiring $(Z^+, \max, \min)$, $I_n = \{1, 2, 3, ..., n\}$ is a k-ideal for any $n \in Z^+$ but not a p-ideal. It is interesting to see that in an inversive semiring S, $E^+(S)$ is a p-ideal. In fact, any full ideal of an inversive semiring is a p-ideal.

Following result was proved in [4, 5].

Proposition 1.2. *In an inversive semiring S an ideal I is a p-ideal if and only if* $I = I + E^+(S)$.

In search of the smallest p-ideal containing a given ideal of a semiring the following definition was given in [4].

Definition 1.3. For any subsemiring R of a semiring S, we define,

$$\hat{R} = \{x \in S \mid a + nx = (n + 1)x, \text{ for some } n \in \mathbb{N}, a \in R\}.$$

Proposition 1.4. [4] *For any two ideals I, J of a semiring S we see that,* $\hat{I}$ *is a p-ideal of S such that* $\hat{\hat{I}} = \hat{I}$; $I \subseteq \hat{I}$ *if* $I \subseteq J$ *then* $\hat{I} \subseteq \hat{J}$; *indeed,* $\hat{I}$ *is the smallest p-ideal of S containing I.*

Note that in case of an *inversive semiring S,* for an ideal I of S,

$$\hat{I} = \{x \in S \mid a + x = 2x \text{ for some } a \in I\}.$$

This leads us to the following result immediately:

Corollary 1.5. [4, 5] *In an inversive semiring S, we have* $\hat{I} = I + E^+(S)$, *for any ideal I.*

2. CONGRUENCE INDUCED BY A p-IDEAL

In keeping with the spirit of the concept of p-ideals in a semiring, let us now introduce the concept of a new kind of congruence induced by an ideal in a semiring S with absorbing zero.

Definition 2.1. Let I be an ideal in a semiring S. We define a *relation* ρ_I on S, *induced by* I, as follows: for $a, b \in S$,

$$a\rho_I b \Leftrightarrow \begin{array}{l} na + x + b = (n+1)a \\ mb + y + a = (m+1)b, \end{array}$$

for some $x, y \in I$ and $m, n \in \mathbb{N}$.

Proposition 2.2. ρ_I *(as defined above) is a congruence on* S.

Proof. We first show that ρ_I is an equivalence relation on S. Since S has an absorbing zero, it follows that ρ_I is reflexive; whereas symmetry of ρ_I is obvious from its definition. Towards transitivity, let $a\,\rho_I\,b,\ b\,\rho_I\,c$ for some $a, b, c \in S$. From the definition of ρ_I we have,

(1) $na + x + b = (n+1)a$, and

(2) $mb + y + a = (m+1)b$, for some $x, y \in I$; $m, n \in \mathbb{N}$ and

(3) $rb + z + c = (r+1)b$, and

(4) $sc + w + b = (s+1)c$, for some $z, w \in I$; $r, s \in \mathbb{N}$.

Now, from (1) we have,

$$(1+r)na + (1+r)x + (1+r)b = (1+r)(n+1)a$$

i.e. $(1+r)na + (1+r)x + rb + z + c = (1+r)(n+1)a$ [by (3)]

i.e. $rna + na + x + rx + rb + z + c = (1+r)(n+1)a$

i.e. $r(na + x + b) + na + x + z + c = (nr + n + r + 1)a$

i.e. $r(n+1)a + na + x + z + c = (nr + n + r + 1)a$ [by (1)]

i.e. $(nr + r + n)a + (x + z) + c = (nr + n + r + 1)a$

i.e. $ka + (x + z) + c = (k+1)a$,

where $x + z \in I$ and $k = nr + n + r \in \mathbb{N}$.

Again, we have, from (4),

$$(m+1)sc + (m+1)w + (m+1)b = (m+1)(s+1)c$$

i.e. $msc + sc + mw + w + mb + y + a = (ms + s + m + 1)c$ [by (2)]

i.e. $m(sc + w + b) + sc + w + y + a = (ms + s + m + 1)c$

i.e. $m(s + 1)c + sc + w + y + a = (ms + s + m + 1)c$ [by (4)]

i.e. $tc + (w + y) + a = (t + 1)c$,

where $w + y \in I$ and $t = ms + s + m \in \mathbb{N}$.

Consequently, we have $a\rho_I c$ i.e., ρ_I is a transitive relation and hence an equivalence relation on S. We now show that, actually ρ_I is a congruence relation on S. Let, for some $a, b, c, d \in S$, $a\rho_I b$ and $c\rho_I d$ hold. Then we have,

(5) $na + x + b = (n + 1)a$, and

(6) $mb + y + a = (m + 1)b$, for some $x, y \in I$; $m, n \in \mathbb{N}$ and

(7) $sc + w + d = (s + 1)c$, and

(8) $td + z + c = (t + 1)d$, for some $w, z \in I$; $s, t \in \mathbb{N}$.

It is easy to see that,

$$k(a + c) + (x + w) + (b + d) = (k + 1)(a + c),$$

where $x + w \in I$ and $k = \max(n, s) \in \mathbb{N}$, and also,

$$r(b + d) + (z + y) + (a + c) = (r + 1)(b + d),$$

where $y + z \in I$ and $r = \max(m, t) \in \mathbb{N}$, which together imply that, $(a + c)\rho_I(b + d)$ hold.

We further see that, for some $c \in S$, we have from (5) and (6) that,

$$nac + xc + bc = (n + 1)ac \text{ and}$$

$$mbc + yc + ac = (m + 1)bc,$$

where $xc, yc \in I$ as I is an ideal, showing that, $(ac)\rho_I(bc)$ holds. In a similar manner, it can be also shown that, $(ca)\rho_I(cb)$ holds, whence ρ_I is a congruence on S.

Remark. It is interesting to note that, when I is a p-ideal, elements of I are ρ_I related only to elements of I i.e., $b \in I$ with $a\rho_I b$ implies that $a \in I$. Indeed, from $na + x + b = (n + 1)a$, for some $x \in I$ and $n \in \mathbb{N}$ [as in (5)], with $b \in I$ we have, $(x + b) \in I$, whence, as I is a p-ideal we get, $a \in I$. So we see that ρ_I saturates the p-ideal I. However, we point out that, for any two elements $a, b \in I$, it is not mandetory that $a\rho_I b$ holds;

i.e. a p-ideal I is being actually partitioned into disjoint ρ_I classes. Indeed, we have the following:

Proposition 2.3. *I constitutes a ρ_I class if and only if I is a subring of S.*

Proof. Let I be a subring of S, and let $a, b \in I$ be any two elements of I. Then choosing $x = a - b \in I$ and $y = b - a \in I$ it can be easily seen that the conditions (5) and (6) of Proposition 2.2 are automatically satisfied, for any $m, n \in \mathbb{N}$ whence $a\rho_I b$ holds, showing thereby that, I is a single ρ_I class.

Conversely, let I be a ρ_I class, i.e., for all $a, b \in I$, $a\rho_I b$ holds. Since I is an ideal of S, we have $0 \in I$, which indicates, $a\rho_I 0$ must hold good; whence for any $a \in I$ we get that there exists some $y \in I$ such that $a + y = 0$, showing that I is a subring of S.

Proposition 2.4. For any ideal I of a semiring S, $S/I = S/\hat{I}$.

Proof. Since $I \subseteq \hat{I}$, $a\rho_I b \Rightarrow a\rho_{\hat{I}} b$ for any $a, b \in S$.

Conversely, let $a\rho_{\hat{I}} b$ hold for some $a, b \in S$. Then,

(1) $na + x + b = (n + 1)a$ and

(2) $mb + y + a = (m + 1)b$

for some $x, y \in \hat{I}$ and $m, n \in \mathbb{N}$. Now, since $x, y \in \hat{I}$, we have that

(3) $kx + i_1 = (k + 1)x$ and

(4) $ly + i_2 = (l + 1)y$

for some $i_1 + i_2 \in I$ and $k, l \in \mathbb{N}$. Then from (3), we have that,

$$kx + i_1 + (k + 1)b + n(k + 1)a = (k + 1)x + (k + 1)b + n(k + 1)a$$

i.e. $k(na + x + b) + na + i_1 + b = (k + 1)(na + x + b)$

i.e. $k(n + 1)a + na + i_1 + b = (k + 1)(n + 1)a$ [from (1)]

i.e. $pa + i_1 + b = (p + 1)a$ ...(1)

where $p = kn + k + n \in \mathbb{N}$.

Now from (4), we have that,

$$ly + i_2 + (l + 1)a + m(l + 1)b = (l + 1)y + (l + 1)a + m(l + 1)b$$

i.e., $l(mb + y + a) + mb + i_2 + a = (l + 1)(mb + y + a)$

i.e., $l(m + 1)b + mb + i_2 + a = (l + 1)(m + 1)b$ [from (2)]

i.e., $qb + i_2 + a = (q + 1)b$ ...(*ii*)

where $q = lm + l + m \in \mathbb{N}$.

Hence from (i) and (ii), we conclude that $a\rho_I b$ holds.

Proposition 2.5. For any ideal I of a semiring S and for any $a\rho_I \in P^+(S/I)$ there exists $i \in P^+(S)$ such that $b = a + i \in P^+(S)$ and $a\rho_I = b\rho_I$.

Proof. Let $a\rho_I \in P^+(S/I)$. Then, for some $n \in \mathbb{N}$, $na\rho_I = (n + 1)a\rho_I$. Hence we have the following:

$$k(na) + i + (n + 1)a = (k + 1)na, \quad l(n + 1)a + j + na = (l + 1)(n + 1)a$$

for some $k, l \in \mathbb{N}$ and $i, j \in I$. Suppose, $p = \max. \{l, k\}$. Then,

(1) $p(na) + i + (n + 1)a = (p + 1)na$ and

(2) $p(n + 1)a + j + na = (p + 1)(n + 1)a$

Now from (1), we have that,

$$(pn + n)a + (a + i) = (pn + n)a \quad ...(iii)$$

i.e., $(pn + n)a + (a + i) + (pn + n)i = (pn + n)a + (pn + n)i$

i.e., $(pn + n)(a + i) + (a + i) = (pn + n)(a + i)$

i.e., $(pn + n + 1)(a + i) = (pn + n)(a + i)$

i.e. $b = a + i \in P^+(S)$.

From (2), we have that,

$$(pn + n + p)a + j = (pn + n + p + 1)a$$

i.e., $(pn + n + p)a + j + (a + i) = (pn + n + p + 1)a$ [from (iii)]

i.e., $ma + j + b = (m + 1)a$ ...(α)

where $m = pn + n + p \in \mathbb{N}$.

Moreover, $(a + i) + i + a = 2(a + i)$, i.e., $b + i + a = 2b$ ...(β)

(α) and (β) together imply that $a\rho_I = b\rho_I$.

Remark 2.6. Since for any $a \in P^+(S)$, $a\rho_I$ $P^+(S/I)$, so from Proposition 2.5., $P^+(S/I) = \{a\rho_I : a \in P^+(S)\}$.

Proposition 2.7. For ay ideal I of a semiring S, $P = \bigcup_{a \in P^+(S)} a\rho_I$ is a p-ideal of S, contained in $\hat{I}$.

Proof. For any, $p, q \in P$, $\exists a, b\ P^+(S)$ such that $p \in a\rho_I$ and $q \in b\rho_I$. Then, $p + q \in (a + b)\rho_I$ and $xpy \in (xay)\rho_I$ for any $x, y \in S$ and hence P is an ideal of S.

Let $x \in \hat{P}$ be arbitrary. Then, $\exists b \in P$ and $m \in \mathbb{N}$ such that $mx + b = (m + 1)x$. Hence, $mx\rho_I + b\rho_I = (m + 1)x\rho_I$ whence $x\rho_I \in P^+(S/I)$ (since $P^+(S/I)$ is a p-ideal of the semiring S/I and $b\rho_I \in P^+(S/I)$). Hence $x \in P$. So, P is a p-ideal of S. Now $b \in P = \bigcup_{a \in P^+(S)} a\rho_I \Rightarrow$ $\exists a \in P^+(S)$ such that $b \in a\rho_I$. Thus, there exist $k \in \mathbb{N}$ and $j \in I$ such that $kb + j + a = (k + 1)b$. Thus $b \in P$ (since P is p-ideal of S and $P^+(S) \subseteq I$).

Proposition 2.8. *In an inversive semiring S, the ρ_I classes of a full k-ideal I, and hence ρ_I classes of a p-ideal I which is also a k-ideal, are precisely the idempotents of S/ρ_I.*

Proof. We recall that in an inversive semiring S, any full ideal is a p-ideal Let I be a full k-ideal in S. We show that,

$$E^+(S/\rho_I) = I/\rho_I.$$

Indeed, let $a\rho_I \in E^+(S/\rho_I)$, for some $a \in S$, so that, $a\rho_I 2a$ gives $3a + x = 2a$ and $3a + y = 4a$ for some $x, y \in I$ where the second relation gives $a + y = 2a$ on addition of adequate number of a' in both sides, giving that, $a \in I$ i.e., $a\rho_I \in I/\rho_I$ so that,

$$E^+(S/\rho_I) \subseteq I/\rho_I.$$

Conversely, we assume $a\rho_I \in I/\rho_I$ so that, $a \in I$ which is a full k-ideal of S. Then $a\rho_I 2a$ holds; indeed, I being a full k-ideal we have $a' \in I$ so that, $a + a' + 2a = 2a$ and $2a + a + a = 4a$ establishes our claim; hence $a\rho_I \in E^+(S/\rho_I)$ i.e.,

$$I/\rho_I \subseteq E^+(S/\rho_I).$$

Hence the result follows.

3. BOURNE p-CONGRUENCE

Definition 3.1. Let I be an ideal in a semiring S. We define a *relation* ρ_I on S, *induced by* I, as follows: for $a, b \in S$,

$$\begin{aligned} a\rho_I b \Leftrightarrow na + x_1 + b &= (n + 1)a + x_2 \\ mb + y_1 + a &= (m + 1)b + y_2, \end{aligned}$$

for some $x_1, x_2, y_1, y_2 \in I$ and $m, n \in \mathbb{N}$.

Remark 3.2. For any ideal I of a semiring S, if $\equiv_I$ denotes the Bourne relation [1] on S, then $\equiv_I \subseteq \rho_I$. Indeed, let for any $a, b \in S$, $a \equiv_I b$. Then, $\exists i, j \in I$ such that $a + i = b + j$ and so by adding a and b separately to both the sides, we get, $a + j + b = 2a + i$ and $b + i + a = 2b + j$ whence $a\rho_I b$ holds.

Proposition 3.3. *ρ_I (as defined above) is a congruence on S.*

Proof. We first show that ρ_I is an equivalence relation on S. Since S has an absorbing zero, it follows that, ρ_I is reflexive; whereas symmetry of ρ_I is obvious from its definition. Towards transitivity, let $a\,\rho_I\,b$, $b\,\rho_I\,c$ for some $a, b, c \in S$. From the definition of ρ_I we have,

(1) $na + x_1 + b = (n + 1)a + x_2$, and

(2) $mb + y_1 + a = (m + 1)b + y_2$, for some $x_1, x_2, y_1, y_2 \in I$; $m, n \in \mathbb{N}$ and

(3) $rb + z_1 + c = (r + 1)b + z_2$, and

(4) $sc + w_1 + b = (s + 1)c + w_2$, for some $z_1, z_2, w_1, w_2 \in I$; $r, s \in \mathbb{N}$.

Now from (1) we have that,

$$(r + 1)na + (r + 1)x_1 + (r + 1)b + z_2 = (r + 1)(n + 1)a + (r + 1)x_2 + z_2$$

i.e., $(r + 1)na + (r + 1)x_1 + rb + z_1 + c = (r + 1)(n + 1)a + (r + 1)x_2 + z_2$ [from 3]

i.e., $na + r(na + x_1 + b) + z_1 + x_1 + c = (r + 1)(n + 1)a + (r + 1)x_2 + z_2$

i.e., $na + r[(n + 1)a + x_2] + x_1 + z_1 + c = (r + 1)(n + 1)a + (r + 1)x_2 + z_2$ [from 1]

i.e., $(rn + n + r)a + [rx_2 + x_1 + z_1] + c = (rn + n + r + 1)a + [(r + 1)x_2 + z_2]$

i.e., $pa + i_1 + c = (p + 1)a + i_2$ ... (α)

where $p = rn + n + r \in \mathbb{N}$ and $i_1 = rx_2 + x_1 + z_1$, $i_2 = (r + 1)x_2 + z_2 \in I$.

Again, from (4) we have that,

$$(m + 1)sc + (m + 1)w_1 + (m + 1)b + y_2 = (m + 1)(s + 1)c + (m + 1)w_2 + y_2$$

i.e., $(m + 1)sc + (m + 1)w_1 + mb + y_1 + a = (m + 1)(s + 1)c + (m + 1)w_2 + y_2$ [from 2]

i.e., $(m+1)sc + (m+1)w_1 + mb + y_1 + a = (m+1)(s+1)c + (m+1)w_2 + y_2$

i.e., $sc + m(sc + w_1 + b) + w_1 + y_1 + a = (m+1)(s+1)c + (m+1)w_2 + y_2$

i.e., $sc + m[(s+1)c + w_2] + w_1 + y_1 + a = (m+1)(s+1)c + (m+1)w_2 + y_2$ [from 4]

i.e., $(ms + m + s)c + [mw_2 + w_1 + y_1] + a = (ms + m + s + 1)c + [(m+1)w_2 + y_2]$

i.e., $qc + j_1 + a = (q+1)c + j_2$ ... (β)

where $q = ms + m + s \in \mathbb{N}$ and $j_1 = mw_2 + w_1 + y_1$, $j_2 = (m+1)w_2 + y_2 \in I$.

Clearly (α) and (β) together imply that $a\rho_I c$ holds i.e., the relation ρ_I is transitive. So, ρ_I is an equivalence relation. We now show that, actually ρ_I is a congruence relation of S. Let, for some $a, b, c, d \in S$, $a\rho_I b$ and $c\rho_I d$ hold. Then we have,

(5) $na + x_1 + b = (n+1)a + x_2$, and

(6) $mb + y_1 + a = (m+1)b + y_2$, for some $x_1, x_2, y_1, y_2 \in I$; $m, n \in \mathbb{N}$ and

(7) $sc + w_1 + d = (s+1)c + w_2$, and

(8) $td + z_1 + c = (t+1)d + z_2$, for some $w_1, w_2, z_1, z_2 \in I$; $s, t \in \mathbb{N}$.

It is easy to see that,

$$k(a+c) + (x_1 + w_1) + (b+d) = (k+1)(a+c) + (x_2 + w_2),$$

where $x_1 + w_1, x_2 + w_2 \in I$ and $k = \max(n, s) \in \mathbb{N}$, and also,

$r(b+d) + (y_1 + z_1) + (a+c) = (r+1)(b+d) + (y_2 + z_2)$,

where $y_1 + z_1, y_2 + z_2 \in I$ and $r = \max(m, t) \in \mathbb{N}$, which together imply that, $(a+c)\rho_I(b+d)$ hold.

we further see that, for some $c \in S$, we have from (5) and (6) that,

$$nac + x_1c + bc = (n+1)ac + x_2c \quad \text{and}$$

$$mbc + y_1c + ac = (m+1)bc + y_2c,$$

where $x_1c, x_2c, y_1c, y_2c \in I$ as I is an ideal, showing that, $(ac)\rho_I(bc)$ holds. In a similar maner, it can be also shown that, $(ca)\rho_I(cb)$ holds, whence ρ_I is a congruence on S.

Proposition 3.4. For an ideal I of a semiring S, $P = \{a \in S : a\rho_I \in P^+(S/I)\}$ is a p-ideal of S containing I. also, $\bar{\hat{I}} = \bar{P}$.

Proof. Let $a, b \in P$. Then $\exists n, m \in \mathbb{N}$ such that $na\rho_I = (n+1)a\rho_I$ and $mb\rho_I = (m+1)b\rho_I$ whence $k(a+b)\rho_I = (k+1)(a+b)\rho_I$ where $k = \max.\{n, m\}$. thus $a + b \in P$. For any $x \in S$, $na\rho_I = (n+1)a\rho_I \Rightarrow nxa\rho_I = (n+1)xa\rho_I$, $nax\rho_I = (n+1)ax\rho_I \Rightarrow xa, ax \in P$. So, P is an ideal of S.

Let $x \in S$ such that $nx + a = (n+1)x$ for some $a \in P$ and $n \in \mathbb{N}$. Then, $nx\rho_I + a\rho_I = (n+1)x\rho_I \Rightarrow x\rho_I \in P^+(S/I)$ (since, $P^+(S/I)$ is a p-ideal of S/I and $a\rho_I \in P^+(S/I)) \Rightarrow x \in P$. So, P is a p-ideal of S.

Since, $0 \in I$, for any $i \in I$, $i + i + 0 = 2i + 0$ and $0 + i + i = 0 + 2i$ which together imply that, $i\rho_I 0$ holds for all $i \in I$. Hence, $I \subseteq 0\rho_I$. Clearly, $\bigcup_{a \in P^+(S)} a\rho_I \subseteq P$; so, in particular $0\rho_I \subseteq P$. Consequently, $I \subseteq P$. So, $\bar{\hat{I}} \subseteq \bar{\hat{P}} = \bar{P}$ (since, P is a p-ideal of S). Conversely, let $x \in \bar{P}$. Then, $x + p = q$ for some $p, q \in P$. Now, $p \in P \Rightarrow np\rho_I = (n+1)p\rho_I$ for some $n \in \mathbb{N} \Rightarrow k(n+1)p + i + np = (k+1)(n+1)p + j$ for some $i, j \in I$ and $k \in \mathbb{N} \Rightarrow (kn + k + n)p + i + (kn + k + n)j = (kn + k + n + 1)p + j + (kn + k + n)j \Rightarrow (kn + k + n)(p + j) + i = (kn + k + n + 1)(p + j) \Rightarrow p + j \in \hat{I} \Rightarrow p \in \hat{I}$. Similarly, $q \in \hat{I}$. Thus, $x + p = q \Rightarrow x \in \hat{I}$.

Proposition 3.5. If I is a p-ideal of S, then $S/I = S/\bar{I}$ with $P^+(S/I) = \{0\rho_I\}$.

Proof. Since, $I \subseteq \bar{I}$, it is immediate to write that $a\rho_I b \Rightarrow a\rho_{\bar{I}} b$ for any, $a, b \in S$. Conversely, let $a, b \in S$ such that $a\rho_{\bar{I}} b$ holds. Then,

(1) $na + x_1 + b = (n+1)a + x_2$,

(2) $mb + y_1 + a = (m+1)b + y_2$,

for some $x_1, x_2, y_1, y_2 \in \bar{I}$; m, n $\mathbb{N}$. Moreover,

(3) $x_1 \in \bar{I} \Rightarrow x_1 + i_1 = i'_1$,

(4) $x_2 \in \bar{I} \Rightarrow x_2 + i_2 = i'_2$,

(5) $y_1 \in \bar{I} \Rightarrow y_1 + j_1 = j_1'$,

(6) $y_2 \in \bar{I} \Rightarrow y_2 + j_2 = j_2'$,

for some $i_k, i_k', j_k, j_k' \in I (k = 1, 2)$; $m, n \in \mathbb{N}$. Now, from (1), (3), (4) we have that, $na + (x_1 + i_1) + i_2 + b = (n + 1)a + (x_2 + i_2) + i_1 \Rightarrow na + \left(i_1' + i_2\right) + b = (n + 1)a + \left(i_2' + i_1\right)$ $\Rightarrow na + i + b = (n + 1)a + i'$ ... (α), where $i = i_1' + i_2$, $i' = i_2' + i_1 \in I$.

Again, from (2), (5), (6) we have that,

$mb + (y_1 + j_1) + j_2 + a = (m + 1)b + (y_2 + j_2) + j_1 \Rightarrow mb + \left(j_1' + j_2\right) + a = (m + 1)b + \left(j_2' + j_1\right) \Rightarrow mb + j + a = (m + 1)b + j'$ ... (β), where $j = j_1' + j_2$, $j' = j_2' + j_1 \in I$.

Thus, from (α) and (β), $a\rho_I b$ holds.

Now, from Proposition 3.3 $\bar{I} = \bar{P}$ (since, I is a p-ideal). Consequently, $\bar{I} \subseteq \{a \in S : a\rho_{\bar{I}} \in P^+(S/\bar{I})\}$ (Proposition 3.) $= \{a \in S : a\rho_I \in P^+(S/I)\} = P \subseteq \bar{P} = \bar{I}$ whence $\bar{I} = P$. It is easy to show that $0\rho_I$ is a k-ideal of S. Hence, $P = \bar{I} \subseteq 0\rho_I \subseteq P \Rightarrow P = 0\rho_I$ $\Rightarrow P^+(S/I) = \{0\rho_I\}$.

Proposition 3.6. If I is a maximal p-ideal of a commutative semiring S, such that $\bar{I} \neq S$, then S/I is a p-semifield.

Proof. Since, I is a maximal p-ideal and $\bar{I} \neq S$, so $I = \bar{I} = 0\rho_I$ and thus $P^+(S/I) = \{0\rho_I\} = \{I\}$. Let J be a p-ideal of S/I such that $P^+(S/I) \subseteq J \subseteq S/I$. Let us write, $J_0 = \{a \in S : a\rho_I \in J\}$. Clearly, J_0 is p-ideal of S. Moreover, since $\{I\} = P^+(S/I) \subseteq J$, so $I \subseteq J_0$. Then, by the maximality of I as p-ideal, $J_0 = S$. So, $J = S/I$. Hence, S/I is a p-semifield.

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MINIMAL STRUCTURES, *M*-CONTINUITY FOR MULTIFUNCTIONS AND BITOPOLOGICAL SPACES

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ABSTRACT : By using *M*-continuity between *m*-spaces, we establish the unified theory of several modified forms of continuity for multifunctions between bitopological spaces.

Key words and phrases : *m*-structure, *m*-open (*i, j*)-*M*-continuous, *M*-continuous, bitopological space, multifunction.
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1. INTRODUCTION

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researches of generalizations of continuity in topological spaces and bitopological spaces. By using these sets, many authors introduced and studied various types of modifications of continuity for functions and multifunctions in topological spaces and bitopological spaces. The notion of irresolute functions was introduced b Crossley and Hildebrand [8]. This notion was extended to multifunctions by Ewert and Lipski [11] and studied in [40] and [43]. The notion of preirresolute multifunctions is introduced in [41] and studied in [25]. The notion of α-irresolute multifunctions is introduced by Noiri and Nasef [34] and studied in [44]. Recently, Abd El-Monsef and Nasef [3] introduced and studied the notion of γ-irresolute multifunctions. Furthermore, β-irresolute multifunctions are studied in [40] and almost irresolute multifunctions are studied in [48].

Maheshwari and Prasad [23] and Bose [6] introduced the concepts of semi-open sets and semi-continuity in bitopological spaces. Jelić [12], Kar and Bhattacharyya [14] and Khedr et al. [16] introduced the concepts of preopen sets and precontinuity in bitopological spaces. The notions of α-open sets (or feebly open sets) and α-continuity (or feeble continuity) in bitopological spaces were studied in [13], [31] and [18]. The notions of semi-preopen sets and semi-precontinuity in bitopological spaces were studied in [16]. The notion of irresolute functions in bitopological spaces is introduced and studied in [24]. In [32] Nasef studied almost quasi-continuous functions in bitopological spaces. Some generaliztions of irresolute functions in bitopological spaces are studied in [18], [31] and [17].

Recently, the present authors [45], [46] introduced and investigated the notions of minimal structures, m-spaces, m-continuity and M-continuity for functions. Moreover, in [36], we extended the concept of M-continuity to functions in bitoopological spaces. In the present paper, we introduce multifunctions between bitopological spaces called (i, j)- upper/lower M-continuous. The multifunctions turn out generalizations of the following functions and multifunctions:

(1) some irresolute multifunctions in topological spaces,

(2) modifications of continuous functions in bitopological spaces,

(3) (i, j)-M-continuous functions in bitopological spaces.

We present some characterizations and properties of (i, j)-upper/lower M-continuous multifunctions in bitopological spaces. In the last section, we introduce several new types of (i, j)-upper/lower M-continuous multifunctions in bitopological spaces.

2. PRELIMINARIES

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$(resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$). A subset A is said to be *δ-open* [50] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be *α-open* [33] (resp. *semi-open* [21], *preopen* [27], *β-open* [1], *b-open* [4] or *γ-open* [9]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, γ-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$).

Definition 2.2. [The complement of a semi-open (resp. preopen, α-open, β-open, γ-open) set is said to be *semi-closed* [7] (resp. *preclosed* [10], *α-closed* [28], *β-closed* [1], *γ-closed* [9]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, γ-closed) sets of X containing A is called the *semi-closure* [7] (resp. *preclosure* [10], *α-closure* [28], *β-closure* [2], *γ-closure* [9] of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, ${}_\gamma\mathrm{Cl}(A)$).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, γ-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *γ-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, ${}_\gamma\mathrm{Int}(A)$).

Throughout the present paper (X, τ) and (Y, σ) always denote topological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively. A point $x \in X$ is called an *(i, j)-θ-closure point* [15] (resp. *(i, j)-δ-contact point* [5]) of a subset A if $j\mathrm{Cl}(U) \cap A \neq \emptyset$ (resp. $i\mathrm{Int}(j\mathrm{Cl}(U)) \cap A \neq \emptyset$) for every τ_i-open set U of X containing x. The set of all (i, j)-θ-closure (resp. (i, j)-δ-contact) points of A is denoted by $(i, j)\mathrm{Cl}_\theta(A)$ (resp. $(i, j)\mathrm{Cl}_\delta(A)$). If $A = (i, j)\mathrm{Cl}_\theta(A)$ (resp. $A = (i, j)\mathrm{Cl}_\delta(A)$), then A is called *(i, j)-θ-closed* (resp. *(i, j)-δ-closed*). The complement of an (i, j)-θ-closed (resp. (i, j)-δ-closed) set is said to be *(i, j)-θ-open* (resp. *(i, j)-δ-open*). It is proved in [15] that the family of all (i, j)-θ-open (resp. (i, j)-δ-open) sets, denoted by $(i, j)\theta(X)$ (resp. $(i, j)\delta(X)$), is a topology for X. A subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-regular open if $A = i\mathrm{Int}(j\mathrm{Cl}(A))$.

For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

a) *upper-irresolute* [11] (resp. *upper-preirresolute* [41], *upper-α-irresolute* [34], *upper-β-irresolute* [42], *upper-γ-irresolute* [3], *upper almost irresolute* [48]) if $F^+(V)$ is semi-open (resp. preopen, α-open, β-open, γ-open, β-open) for each semi-open (resp. preopen, α-open, β-open, γ-open, semi-open) set V of (Y, σ),

(b) *lower-irresolute* [11] (resp. *lower-preirresolute* [41], *lower-α-irresolute* [34], *lower-β-irresolute* [42], *lower-γ-irresolute* [3], *lower almost irresolute* [48]) if $F^-(V)$ is semi-open (resp. preopen, α-open, β-open, τ-open, β-open) for each semi-open (resp. preopen, α-open, β-open, γ-open, semi-open) set V of (Y, σ).

3. MINIMAL STRUCTURES AND *M*-CONTINUITY

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [45], [46] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. Then the families SO(X), PO(X), $\alpha(X)$, $\beta(X)$ and $\gamma(X)$ are all m-structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [26] as follows:

(1) $m\mathrm{Cl}(A) = \cap\{F : A \subset F,\ X - F \in m_X\}$,

(2) $m\mathrm{Int}(A) = \cup\{U : U \subset A,\ U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$), then we have

(a) $m\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{A})$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A$, ${}_\beta\mathrm{Cl}(A)$, ${}_\gamma\mathrm{Cl}(A)$),

(b) $m\mathrm{Int}(A) = \mathrm{Int}(\mathrm{A})$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, ${}_\gamma\mathrm{Int}(A)$).

Lemma 3.1. (Maki et al. [26]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $m\mathrm{Cl}(X - A) = X - m\mathrm{Int}(A)$ *and* $m\mathrm{Int}(X - A) = X - m\mathrm{Cl}(A)$,

(2) *If* $(X - A) \in m_X$, *then* $m\mathrm{Cl}(A) = A$ *and if* $A \in m_X$, *then* $m\mathrm{Int}(A) = A$,

(3) $m\mathrm{Cl}(\emptyset) = \emptyset$, $m\mathrm{Cl}(X) = X$, $m\mathrm{Int}(\emptyset) = \emptyset$ *and* $m\mathrm{Int}(X) = X$,

(4) *If* $A \subset B$, *then* $m\mathrm{Cl}(A) \subset m\mathrm{Cl}(B)$ *and* $m\mathrm{Int}(A) \subset m\mathrm{Int}(B)$,

(5) $A \subset m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(A) \subset A$,

(6) $m\mathrm{Cl}(m\mathrm{Cl}(A)) = m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(A)) = m\mathrm{Int}(A)$.

Lemma 3.2. (Popa and Noiri [45]). *Let (X, m_X) be an m-space and A a subset of X. Then* $x \in m\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing* x.

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathscr{B}$ [26] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3. (Popa and Noiri [47]). *Let (X, m_X) be an m-space and m_X have property $\mathscr{B}$. Then for a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $m\mathrm{Int}(A) = A$,

(2) *A is m-closed if and only if* $m\mathrm{Cl}(A) = A$,

(3) $m\mathrm{Int}(A) \in m_X$ *and* $m\mathrm{Cl}(A)$ *is m-closed.*

Definition 3.4. A multifunction $F : (X{,}s\ m_X) \to (Y, m_Y)$ is said to be

(1) *upper M-continuous* [35] if for each $x \in X$ and each $V \in m_Y$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower M-continuous* [35] if for each $x \in X$ and each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Theorem 3.1. (Noiri and Popa [35] *For a multifunction $F : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathscr{B}$, the following properties are equivalent:*

(1) *F is upper M-continuous;*

(2) $F^+(V) = m\mathrm{Int}(F^+(V))$ *for every* $V \in m_Y$;

(3) $F^-(K) = m\mathrm{Cl}(F^-(K))$ *for every* m_Y*-closed set* K *of* Y;

(4) $m\mathrm{Cl}(F^-(B)) \subset F^-(m\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^+(m\mathrm{Int}(B)) \subset m\mathrm{Int}(F^+(B))$ *for every subset* B *of* Y.

Corollary 3.1. *Let* m_X *and* m_Y *be m-structures with property* $\mathscr{B}$. *For a multifunction* F: $(X, m_X) \to (Y, m_Y)$, *the following properties are equivalent:*

(1) *F is upper M-continuous;*

(2) $F^+(V)$ *is* m_X*-open for every* V m_Y;

(3) $F^-(K)$ *is* m_X*-closed for every* m_Y*-closed set* K *of* Y.

Proof. The proof follows from Theorem 3.1 and Lemma 3.3.

Theorem 3.2. (Noiri and Popa [35]) *For a multifunction* $F : (X, m_X) \to (Y, m_Y)$, *where* m_Y *has property* $\mathscr{B}$, *the following properties are equivalent:*

(1) *F is lower M-continuous;*

(2) $F^-(V) = m\mathrm{Int}(F^-(V))$ *for every* $V \in m_Y$;

(3) $F^+(K) = m\mathrm{Cl}(F^+(K))$ *for every* m_Y*-closed set* K *of* Y;

(4) $m\mathrm{Cl}(F^+(B)) \subset F^+(m\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^-(m\mathrm{Int}(B)) \subset m\mathrm{Int}(F^-(B))$ *for every subset* B *of* Y;

(6) $F(m\mathrm{Cl}(A)) \subset m\mathrm{Cl}(F(A))$ *for every subset* A *of* X.

Corollary 3.2. *Let* m_X *and* m_Y *be m-structures with property* $\mathscr{B}$. *For a multifunction* F : $(X, m_X) \to (Y, m_Y)$, *the following properties are equivalent:*

(1) *F is lower M-continuous;*

(2) $F^-(V)$ *is* m_X*-open for every* $V \in m_Y$;

(3) $F^+(K)$ *is* m_X*-closed for every* m_Y*-closed set* K *of* Y.

Proof. The proof follows from Theorem 3.2 and Lemma 3.3.

4. MINIMAL STRUCTURES AND BITOPOLOGICAL SPACES

First, we recall some definitions of weak forms of open sets in a bitopological space.

Definition 4.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-semi-open* [23] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-preopen* [12] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-α-open* [13] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-semi-preopen* [16] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$,

(5) *faintly semi-open* [29] if $A \subset \tau_2\text{-}\mathrm{Cl}(\tau_1\text{-}\mathrm{Int}(A)) \cup \tau_1\text{-}\mathrm{Cl}(\tau_2\text{-}\mathrm{Int}(A))$.

The family of (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-preopen) sets of (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

Remark 4.1. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then $(i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$ and $(i, j)\mathrm{SPO}(X)$ are all m-structures on X. Hence, if $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$), then we have

(1) $m_X^{ij}\mathrm{Cl}(A) = (i, j)\text{-}\mathrm{sCl}(A)$ [23] (resp. $(i, j)\text{-}\mathrm{pCl}(A)$ [16], $(i, j)\text{-}\alpha\mathrm{Cl}(A)$ [31], $(i, j)\text{-}\mathrm{spCl}(A)$ [16]),

(2) $m_X^{ij}\mathrm{Int}(A) = (i, j)\text{-}\mathrm{sInt}(A)$ (resp. $(i, j)\text{-}\mathrm{pInt}(A)$, $(i, j)\text{-}\alpha\mathrm{Int}(A)$, $(i, j)\text{-}\mathrm{spInt}(A)$).

Remark 4.2. Let (X, τ_1, τ_2) be a bitopological space. Then

(1) Let $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\alpha(X)$). Then by Lemma 3.1 we obtain the results established in Theorem 13 of [23] (resp. Theorem 3.6 of [31]).

(2) Let $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 3.2 we obtain the results established in Theorem 1.15 of [22] (resp. Theorem 3.5 of [16], Theorem 3.5 of [31], Theorem 3.5 of [16]).

Remark 4.3. Let (X, τ_1, τ_2) be a bitopological space. Then

(1) $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$) is an m-structure on X satisfying property $\mathcal{B}$ by Theorem 2 of [23] (resp. Theorem 4.2 of [14] or Theorem 3.2 of [16], Theorem 3.2 of [31], Theorem 3.2 of [16]).

(2) Let $m_X^{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 3.3 we obtain the results established in Theorem 1.13 of [22] (resp. Theorem 3.5 of [16], Theorem 3.6 of [31], Theorem 3.6 of [16]).

5. *M*-CONTINUITY AND BITOPOLOGICAL SPACES

Definition 5.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-irresolute* [24] (resp. *(i, j)-preirresolute* [18], *(i, j)-α-irresolute* or *(i, j)-feebly continuous* [31], *(i, j)-β-irresolute* [42], *(i, j)-δ-continuous* [17], *FS-irresolute* [29] if $f^{-1}(V)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, b)-β-open, (i, j)-δ-open, faintly semi-open) in X for each (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-β-open, (i, j)-δ-open, faintly semi-open) set V of Y.

Definition 5.2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-faintly semi-continuous* [19] (resp. *(i, j)-faintly precontinuous* [19], *(i, j)-faintly β-continuous* [19] if $f^{-1}(V)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-β-open) in X for each (i, j)-θ-open set V of Y.

Definition 5.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-almost quasi-continuous* [30] if $f^{-1}(V)$ is (i, j)-semi-open in X for each (i, j)-regular-open set V of Y.

Definition 5.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper δ-continuous* [20] if $F^+(V) \in (i, j)\delta(X)$ for each $V \in (i, j)\delta(Y)$,

(2) *(i, j)-lower δ-continuous* [20] if $F^-(V) \in (i, j)\delta(X)$ for each $V \in (i, j)\delta(Y)$.

Remark 5.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m^{ij}_X (resp. m^{ij}_Y) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2). In case $m^{ij}_X = (i, j)\delta(X)$ and $m^{ij}_Y = (i, j)\delta(Y)$, a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-upper/lower δ-continuous if and only if a multifunction $F : (X, m^{ij}_X) \to (Y, m^{ij}_Y)$ is upper/lower M-continuous.

Now, we can state the main definition of the present paper as follows:

Definition 5.5. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Let m^{ij}_X (resp. m^{ij}_Y) be an m-structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2). A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-upper/lower M-continuous* if a multifunction $F : (X, m^{ij}_X) \to (Y, m^{ij}_Y)$ is upper/lower M-continuous.

Remark 5.2. (1) The (i, j)-upper/lower δ-continuous multifunction is a particular case of (i, j)-upper/lower M-continuous multifunction.

(2) The multifunction in Definition 5.5 is a generalization of each of the following functions:

(a) multifunctions defined by Definition 2.5,

(b) functions defined by Definitions 5.1, 5.2 and 5.3,

(c) functions defined by Definition 4.3 of [36].

By Definition 5.5, Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2, we obtain the following two theorems.

Theorem 5.1. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m^{ij}_X *(resp.* m^{ij}_Y*) be an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m^{ij}_Y *has property* $\mathcal{B}$*. Then for a multifunction* $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) *F is (i, j)-upper M-continuous;*

(2) $F^+(V) = m^{ij}_X\mathrm{Int}(F^+(V))$ *for every* $V \in m^{ij}_Y$*;*

(3) $F^-(K) = m^{ij}_X\mathrm{Cl}(F^-(K))$ *for every* m^{ij}_Y*-closed set K of Y;*

(4) $m^{ij}_X\mathrm{Cl}(F^-(B)) \subset F^-(m^{ij}_Y\mathrm{Cl}(B))$ *for every sbset B of Y;*

(5) $F^+(m^{ij}_Y\mathrm{Int}(B)) \subset m^{ij}_X\mathrm{Int}(F^+(B))$ *for every subset B of Y.*

Corollary 5.1. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m^{ij}_X *(resp.* m^{ij}_Y*) be an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m^{ij}_X *and* m^{ij}_Y *have property* $\mathcal{B}$*. Then, for a multifunction* $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) *F is (i, j)-upper M-continuous;*

(2) $F^+(V)$ *is* m^{ij}_X*-open for every* m^{ij}_Y*-open set V of Y;*

(3) $F^-(K)$ *is* m^{ij}_X*-closed for every* m^{ij}_Y*-closed set K of Y.*

Proof. This is an immediate consequence of Theorem 5.1 and Corollary 3.1.

Remark 5.3. If $m^{ij}_X = (i, j)\delta(X)$ and $m^{ij}_Y = (i, j)\delta(Y)$, then by Theorem 5.1 and Corollary 5.1 we obtain the results established in Theorem 2.3 of [20].

Theorem 5.2. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m^{ij}_X *(resp.* m^{ij}_Y*) be an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m^{ij}_Y *has property* $\mathcal{B}$*. Then, for a multifunction* $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) *F is (i, j)-lower M-continuous;*

(2) $F^-(V) = m^{ij}_X\mathrm{Int}(F^-(V))$ *for every* m^{ij}_Y*-open set V of Y;*

(3) $F^+(K) = m_X^{ij}\mathrm{Cl}(F^+(K))$ *for every* m_Y^{ij}*-closed set* K *of* Y;

(4) $m_X^{ij}\mathrm{Cl}(F^+(B)) \subset F^+(m_Y^{ij}\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^-(m_Y^{ij}\mathrm{Int}(B)) \subset m_X^{ij}\mathrm{Int}(F^-(B))$ *for every subset* B *of* Y;

(6) $F(m_X^{ij}\mathrm{Cl}(A)) \subset m_Y^{ij}\mathrm{Cl}(F(A))$ *for every subset* A *of* X.

Corollary 5.2. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m_X^{ij} *(resp.* m_Y^{ij}*) be an m-structure on* X *(resp.* Y*) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m_X^{ij} *and* m_Y^{ij} *have property* $\mathscr{B}$. *Then, for a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) F *is* (i, j)*-lower M-continuous;*

(2) $F^-(V)$ *is* m_X^{ij}*-open for every* m_Y^{ij}*-open set* V *of* Y;

(3) $F^+(K)$ *is* m_X^{ij}*-closed for every* m_Y^{ij}*-closed set* K *of* Y.

Remark 5.4. (1) If $m_X^{ij} = (i, j)\delta(X)$ and $m_Y^{ij} = (i, j)\delta(Y)$, then by Theorem 5.2 and Corollary 5.2 we obtain the results established in Theorem 2.2 fo [20].

(2) In case $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by Theorem 5.1 (or Theorem 5.2) and Corollary 5.1 (or Corollary 5.2), we obtain the results established in Theorem 4.1 and Corollary 4.1 of [36]).

6. SOME PROPERTIES OF *(i, j)*-UPPER/LOWER *M*-CONTINUITY

Definition 6.1. For a multifunction $F : (X, m_X) \to (Y, m_Y)$, we define a multifunction $\mathrm{mCl}(F)$: $(X, m_X) \to (Y, m_Y)$ as follows: $(\mathrm{mCl}(F))(x) = \mathrm{mCl}(F(x))$ for each $x \in X$.

Lemma 6.1. *If* $F : (X, m_X) \to (Y, m_Y)$ *is a multifunction, then* $(\mathrm{mCl}(F))^-(V) = F^-(V)$ *for each* m_Y*-open set* V *of* Y.

Proof. Let V be any m_Y-open set and $x \in (\mathrm{mCl}(F))^-(V)$. Then $V \cap (\mathrm{mCl}(F))(x) = V \cap \mathrm{mCl}(F(x)) \neq \emptyset$. Therefore, there exists $y \in V \cap \mathrm{mCl}(F(x))$. Since $y \in \mathrm{mCl}(F(x))$ and $y \in V \in m_Y$, by Lemma 3.2 $V \cap F(x) \neq \emptyset$ and hence $x \in F^-(V)$. Conversely, let $V \in m_Y$ and $x \in F^-(V)$, then $\emptyset \neq F(x) \cap V \subset \mathrm{mCl}(F(x)) \cap V = (\mathrm{mCl}(F))(x) \cap V$ and hence $x \in (\mathrm{mCl}(F))^-(V)$.

Theorem 6.1. *A multifunction* $F : (X, m_X) \to (Y, m_Y)$ *is lower M-continuous if and only if* $\mathrm{mCl}(F)$ *is lower M-continuous.*

Proof. *Necessity.* Suppose that F is lower M-continuous. Let $x \in X$ and V be any m_Y-open set of Y such that $(\mathrm{mCl}(F))(x) \cap V \neq \emptyset$. By Lemma 6.1 we have x $(\mathrm{mCl}(F))^-(V) = F^-(V)$ $= F^-(V)$ and hence $F(x) \cap V \neq \emptyset$. Since F is lower M-continuous, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for eacy $u \in U$. Hence we have $(\mathrm{mCl}(F))(u) \cap V \neq \emptyset$ for each $u \in U$. This shows that $\mathrm{mCl}(F)$ is lower M-continuous.

Sufficiency. Suppose that $\mathrm{mCl}(F)$ is lower M-continuous. Let $x \in X$ and V be any m_Y-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, by Lemma 6.1 we have $x \in F^-(V) = (\mathrm{mCl}(F))^-(V)$ and hence $(\mathrm{mCl}(F))(x) \cap V \neq \emptyset$. Since $\mathrm{mCl}(F)$ is lower M-continuous, there exists $U \in m_X$ containing x such that $(\mathrm{mCl}(F))(u) \cap V \neq \emptyset$ for each $u \in U$. By Lemma 6.1, we have $u \in (\mathrm{mCl}(F))^-(V) = F^-(V)$ for each $u \in U$. Thus, we have $F(u) \cap V \neq \emptyset$ for each $u \in U$. Thus, F is lower M-continuous.

Remark 6.1. Let (X, τ) and (Y, σ) be topological spaces. if $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$ and $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$, $\gamma(Y)$), then by Theorem 6.1 we obtain the result established in Theorem 2 of [43] (resp. theorem 3.5 of [41] and Theorem 6 of [25], Theorem 6 of [44], Theorem 3.6 of [40], Theorem 3.5 of [3]).

Corollary 6.1. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces. Let* m^{ij}_X *(resp.* m^{ij}_Y*) be an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*). Then, a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is (i, j)-lower M-continuous if and only if mCl(F)* $: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is (i, j)-lower M-continuous.*

Proof. This follows from Definition 5.5 and Theorem 6.1.

Definition 6.2. An m-space (X, m_X) is said to be *m-compact* [35] if every cover of X by m_X-open sets has a finite subcover. A subset K of (X, m_X) is said to be *m-compact* [44] if every cover of K by m_X-open sets has a finite subcover.

Theorem 6.2. (Noiri and Popa [35]) *Let* (Y, m_Y) *be an m-space and* m_Y *an m-structure with property* $\mathfrak{B}$. *If* $F : (X, m_X) \to (Y, m_Y)$ *is an upper M-continuous multifunction such that F(x) is m-compact for each* $x \in X$ *and K is an m-compact set of X. then F(K) is m-compact.*

Definition 6.3. Let (X, τ_1, τ_2) be a bitopological space and m^{ij}_X an m-structure determined by τ_1 and τ_2. a subset K of X is said to be *(i, j)-m-compact* if K is m^{ij}_X-compact.

Corollary 6.2. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces and* m^{ij}_X *(resp.* m^{ij}_Y*)*

an m-structure on X (resp. Y) determined by τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m_Y^{ij} *has property* $\mathcal{B}$. *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is an (i, j)-upper M-continuous multifunction such that* $F(x)$ *is (i, j)-m-compact for each* $x \in X$ *and K is an (i, j)-m-compact set of X, then* $F(K)$ *is (i, j)-m-compact in Y.*

Proof. This is an immediate consequence of Definitions 5.5 and 6.2 and Thorem 6.2.

Remark 6.2. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Corollary 6.2 we obtain the result established in Theorem 5.2 of [36].

Definition 6.4. A multifunction $F : (X, m_X) \to (Y, m_Y)$ is said to be

(1) *upper M-continuous* at a point $x \in X$ if for each $V\ m_Y$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower M-continuous* at a point $x \in X$ if for each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Definition 6.5. Let (X, m_X) be an m-space and A a subset of X. The *m_X-frontier* of A, denoted by $\mathrm{mFr}(A)$ [46], is defined by $\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A)$.

Theorem 6.3. *The set of all points* $x \in X$ *at which a multifunction* $F : (X, m_X) \to (Y, m_Y)$ *is not upper/lower M-continuous is identical with the union of the* m_X*-frontiers of upper/lower inverse images of* m_Y*-open sets containing/meeting* $F(x)$.

Proof. Let x be a point of (X, m_X) at which F is not upper M-continuous. Then, there exists $V \in m_Y$ containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 3.2, we have $x \in \mathrm{mCl}(X - F^+(V))$. Since $x \in F^+(V)$, we have $x \in \mathrm{mCl}(F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$. Conversely, let $V \in m_Y$ containing $F(x)$ and $x \in \mathrm{mFr}(F^+(V))$. Now, assume that F is upper M-continuous at x, then there exists $U \in m_X$ containing x such that $F(U) \subset V$; hence $U \subset F^+(V)$. Therefore, we obtain $x \in \mathrm{mInt}(F^+(V))$. This is a contradiction. Therefore, F is not upper M-continuous. Since the proof for lower M-continuous multifunctions is similar, it is omitted.

Corollary 6.3. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces and* m_X^{ij} *(reesp.* m_Y^{ij}*) an m-structure on X (resp. Y) determined by* τ_1 *and* τ_2 *(resp.* σ_1 *and* σ_2*), where* m_Y^{ij} *has property* $\mathcal{B}$. *The set of all points* $x \in X$ *which a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is not (i, j)-upper/lower M-continuous is identical with the union of the* m_X^{ij}*-frontiers of upper/lower inverse images of* m_Y^{ij}*-open sets containing/meeting* $F(x)$.

Remark 6.3. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Corollary 6.3 we obtain the result established in Theorem 5.4 of [36].

7. NEW FORMS OF (i, j)-M-CONTINUOUS MULTIFUNCTIONS

There are many modification sof open sets in topological spaces. First, we recall θ-closed sets due to Velicko [50]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for evry open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be *θ-closed* [50]. The complement of a θ-closed set is said to be *θ-open*. The union of all θ-open sets contained in A is called the *θ-interior* of A and is denoted by $\mathrm{Int}_\theta(A)$.

Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. The δ-closure (resp. θ-closure) of A and the δ-interior (resp. θ-interior) of A with respect to τ_i are denoted by ${}_i\mathrm{Cl}_\delta(A)$ (resp. ${}_i\mathrm{Cl}_\theta(A)$) and $i\mathrm{Int}_\delta(A)$ (resp. $i\mathrm{Int}_\theta(A)$). The notions of δ-semiopen sets [39] and δ-preopen sets [49] are generalized in [37] and [38] to the setting of bitopological spaces as follows:

Definition 7.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-δ-semi-open* [37] if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $j = 1, 2$,

(2) *(i, j)-δ-preopen* [38] if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-δ-b-open* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A)) \cup j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-δ-semipreopen* (simply *(i, j)-δ-sp-open*) if there exists an (i, j)-δ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Definition 7.2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-θ-semi-open* if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-θ-preopen* if $A \subset i\mathrm{Int}\,(j\mathrm{Cl}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-θ-b-open* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A)) \cup j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-θ-semipreopen* (simply *(i, j)-θ-sp-open*) if there exists an (i, j)-θ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Let (X, τ_1, τ_2) be a bitopological space. The family of (i, j)-δ-semi-open (resp. (i, j)-δ-preopen, (i, j)-δ-b-open, (i, j)-δ-sp-open, (i, j)-θ-semi-open, (i, j)-θ-preopen, (i, j)-θ-b-open, (i, j)-θ-sp-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\delta\mathrm{SO}(X)$ (resp. $(i, j)\delta\mathrm{PO}(X)$, $(i, j)\delta\mathrm{BO}(X)$, $(i, j)\delta\mathrm{SPO}(X)$, $(i, j)\theta\mathrm{SO}(X)$, $(i, j)\theta\mathrm{PO}(X)$, $(i, j)\theta\mathrm{BO}(X)$, $(i, j)\theta\mathrm{SPO}(X)$).

Remark 7.1. Let (X, τ_1, τ_2) be a bitopological space. The families $(i, j)\delta\mathrm{SO}(X)$, $(i, j)\delta\mathrm{PO}(X)$, $(i, j)\delta\mathrm{BO}(X)$, $(i, j)\delta\mathrm{SPO}(X)$, $(i, j)\theta\mathrm{SO}(X)$, $(i, j)\theta\mathrm{PO}(X)$, $(i, j)\theta\mathrm{BO}(X)$ and $(i, j)\theta\mathrm{SPO}(X)$ are all m-structurs with property $\mathcal{B}$.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ we can define many new types of (i, j)-upper/lower M-continuous multifunctions. For example, in case $m_X^{ij} = (i, j)\delta SO(X)$ (resp. $(i, j)\delta PO(X)$, $(i, j)\delta BO(X)$, $(i, j)\delta SPO(X)$, $(i, j)\theta SO(X)$, $(i, j)\theta PO(X)$, $(i, j)\theta BO(X)$, $(i, j)\theta SPO(X)$ and $m_Y^{ij} = (i, j)\delta SO(Y)$ (resp. $(i, j)\delta PO(Y)$, $(i, j)\delta BO(Y)$, $(i, j)\delta SPO(Y)$, $(i, j)\theta SO(Y)$, $(i, j)\theta PO(Y)$, $(i, j)\theta BO(Y)$, $(i, j)\theta SPO(Y)$), we can define new types of (i, j)-upper/lower M-continuous multifunctions as follows:

Definition 7.3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper/lower δ-semi-irresolute* if $F : (X, (i, j)\delta SO(X)) \to (Y, (i, j)\delta SO(Y))$ is upper/lower M-continuous,

(2) *(i, j)-upper/lower δ-preirresolute* if $F : (X, (i, j)\delta PO(X)) \to (Y, (i, j)\delta PO(Y))$ is upper/lower M-continuous,

(3) *(i, j)-upper/lower δ-b-irresolute* if $F : (X, (i, j)\delta BO(X)) \to (Y, (i, j)\delta BO(Y))$ is upper/lower M-continuous,

(4) *(i, j)-upper/lower δ-sp-irresolute* if $F : (X, (i, j)\delta SPO(X)) \to (Y, (i, j)\delta SPO(Y))$ is upper/lower M-continuous.

Definition 7.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper/lower θ-semi-irresolute* if $F : (X, (i, j)\theta SO(X)) \to (Y, (i, j)\theta SO(Y))$ is upper/lower M-continuous,

(2) *(i, j)-upper/lower θ-preirresolute* if $F : (X, (i, j)\theta PO(X)) \to (Y, (i, j)\theta PO(Y))$ is upper/lower M-continuous,

(3) *(i, j)-upper/lower θ-b-irresolute* if $F : (X, (i, j)\theta BO(X)) \to (Y, (i, j)\theta BO(Y))$ is upper/lower M-continuous,

(4) *(i, j)-upper/lower θ-sp-irresolute* if $F : (X, (i, j)\theta SPO(X)) \to (Y, (i, j)\theta SPO(Y))$ is upper/lower M-continuous.

Conclusion. We can apply the results established in Sections 5 and 6 for the following multifunctions:

(1) the multifunctions defined in Definitions 7.3 and 7.4 and

(2) any (i, j)-upper/lower M-continuous multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by using any m-structures m_X^{ij} and m_Y^{ij}.

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